37

Homogeneous Linear Systems and Their General Solutions

We are now going to restrict our attention further to the standard first-order systems of differential equations that are "linear", with particular attention being given to developing the theory for solving those linear systems that are also "homogeneous". Fortunately, this theory is very similar to that for single linear differential equations developed in chapters 13 and 14. In fact, to some extent, our discussion will be guided by what we already know about general solutions to N^{th}-order linear differential equations. You should also expect to see significant use of a few results from basic linear algebra.

Will we finally actually solve a few systems in this chapter? No, not really, but we will need the theory developed here when we finally do start solving systems in the next chapter.

37.1 Basic Terminology and Notation

Basic Definitions

A standard first-order $N \times N$ system

$$\begin{bmatrix} {x_1}' \\ {x_2}' \\ \vdots \\ {x_N}' \end{bmatrix} = \begin{bmatrix} f_1(t, x_1, x_2, \dots, x_N) \\ f_2(t, x_1, x_2, \dots, x_N) \\ \vdots \\ f_N(t, x_1, x_2, \dots, x_N) \end{bmatrix}$$

is said to be *linear* if and only if each component function f_k can be written as

$$f_k(t, x_1, x_2, \dots, x_N) = p_{k1}x_1 + p_{k2}x_2 + \cdots + p_{kN}x_N + g_k$$

where the p_{kj}'s and g_k's are either constants or functions of t only. If, in addition, all the g_k's are zero, so that our system looks like

$$\begin{bmatrix} {x_1}' \\ {x_2}' \\ \vdots \\ {x_N}' \end{bmatrix} = \begin{bmatrix} p_{11}x_1 + p_{12}x_2 + \cdots + p_{1N}x_N \\ p_{21}x_1 + p_{22}x_2 + \cdots + p_{2N}x_N \\ \vdots \\ p_{N1}x_1 + p_{N2}x_2 + \cdots + p_{NN}x_N \end{bmatrix} , \tag{37.1}$$

then we say our linear system is *homogeneous*. If, on the other hand, we have

$$\begin{bmatrix} x_1{}' \\ x_2{}' \\ \vdots \\ x_N{}' \end{bmatrix} = \begin{bmatrix} p_{11}x_1 + p_{12}x_2 + \cdots + p_{1N}x_N \\ p_{21}x_1 + p_{22}x_2 + \cdots + p_{2N}x_N \\ \vdots \\ p_{N1}x_1 + p_{N2}x_2 + \cdots + p_{NN}x_N \end{bmatrix} + \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_N \end{bmatrix} \tag{37.2}$$

with one or more of the g_k's being nonzero, then we say that the linear system is *nonhomogeneous*.

Matrix/Vector Notation for Linear Systems

Letting

$$\mathbf{P} = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1N} \\ p_{21} & p_{22} & \cdots & p_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ p_{N1} & p_{N2} & \cdots & p_{NN} \end{bmatrix} \quad \text{and} \quad \mathbf{g} = \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_N \end{bmatrix} \quad ,$$

and using the standard rules of matrix multiplication,

$$\mathbf{Px} = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1N} \\ p_{21} & p_{22} & \cdots & p_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ p_{N1} & p_{N2} & \cdots & p_{NN} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} = \begin{bmatrix} p_{11}x_1 + p_{12}x_2 + \cdots + p_{1N}x_N \\ p_{21}x_1 + p_{22}x_2 + \cdots + p_{2N}x_N \\ \vdots \\ p_{N1}x_1 + p_{N2}x_2 + \cdots + p_{NN}x_N \end{bmatrix} \quad ,$$

we can rewrite homogeneous linear system (37.1) and nonhomogeneous linear system (37.2), respectively, as

$$\mathbf{x}' = \mathbf{Px} \tag{37.1$'$}$$

and

$$\mathbf{x}' = \mathbf{Px} + \mathbf{g} \quad . \tag{37.2$'$}$$

Keep in mind that the components of both $\mathbf{P}$ and $\mathbf{g}$ may be functions of t, but not of any of the x_k's. Also note that the system is automatically a regular autonomous system if all the components of $\mathbf{P}$ and $\mathbf{g}$ are constants.

As with "vectors", we will refer to any matrix $\mathbf{P}$ as being a *constant matrix* or a *matrix-valued function* (on some interval) according to whether the components of $\mathbf{P}$ are all constants or can be functions (on the given interval). If, in addition, all the components of $\mathbf{P}$ are continuous functions on some interval, then we will say $\mathbf{P}$ is a *continuous* matrix-valued function on that interval. Let us also agree that (unless otherwise indicated) all components of our matrices and vectors are real-valued. If it seems particularly relevant, we'll explicitly say that a given matrix is *real* or *real-valued* to indicate that its components are real values or real-valued functions.

Expanding on our conventions from the previous chapter, we will, as much as possible, use bold-faced capital letters to denote generic $N \times N$ matrices, with the corresponding lower-case, nonbold letters, suitably subscripted, to denote the corresponding components of the matrix. So if we have a matrix $\mathbf{A}$, then (unless otherwise indicated)

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N1} & a_{N2} & \cdots & a_{NN} \end{bmatrix} \quad .$$

Let us also agree that, when discussing a standard $N \times N$ linear system, every vector and matrix under discussion consisting of a single row or column has N components, and that every matrix under discussion not consisting of a single row or column is a $N \times N$ matrix.

Sets of Vectors

Keep in mind that the $\mathbf{x}$ in the above discussion denotes a vector-valued function on some interval,

$$\mathbf{x}(t) = \left[x_1(t)\ ,\ x_2(t)\ ,\ \dots\ ,\ x_N(t)\right]^{\mathsf{T}} \quad ,$$

Often, in what follows, we will have several such vector-valued functions. When we do, we may use superscripts to distinguish the different vector-valued functions; that is, we will write the set of vector-valued functions as either

$$\left\{\mathbf{x}^1\ ,\ \mathbf{x}^2\ ,\ \dots\ ,\ \mathbf{x}^M\right\} \qquad \text{or} \qquad \left\{\mathbf{x}^1(t)\ ,\ \mathbf{x}^2(t)\ ,\ \dots\ ,\ \mathbf{x}^M(t)\right\}$$

with

$$\mathbf{x}^1(t) = \begin{bmatrix} x_1^1(t) \\ x_2^1(t) \\ \vdots \\ x_N^1(t) \end{bmatrix} \quad , \quad \mathbf{x}^2(t) = \begin{bmatrix} x_1^2(t) \\ x_2^2(t) \\ \vdots \\ x_N^2(t) \end{bmatrix} \quad , \quad \dots \quad \text{and} \quad \mathbf{x}^M(t) = \begin{bmatrix} x_1^M(t) \\ x_2^M(t) \\ \vdots \\ x_N^M(t) \end{bmatrix} \quad .$$

!►*Example 37.1:* *Consider the system*

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x + 2y \\ 5x - 2y \end{bmatrix} \quad .$$

This is easily seen to be a homogeneous linear 2×2 system of differential equations, and since

$$\begin{bmatrix} x + 2y \\ 5x - 2y \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 5 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \quad ,$$

we see that we can rewrite it as either

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 5 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

or even as

$$\mathbf{x}' = \mathbf{P}\mathbf{x}$$

where

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} \qquad \textit{and} \qquad \mathbf{P} = \begin{bmatrix} 1 & 2 \\ 5 & -2 \end{bmatrix} \quad .$$

In this case $\mathbf{P}$ is a constant matrix and, hence, is automatically a continuous 2×2 matrix-valued function over the interval $(-\infty, \infty)$. It is easily verified (see example 35.2 and exercise 35.4) that one pair of solutions $\{\mathbf{x}^1, \mathbf{x}^2\}$ to the above system is given by

$$\mathbf{x}^1(t) = \begin{bmatrix} e^{3t} \\ e^{3t} \end{bmatrix} \qquad \textit{and} \qquad \mathbf{x}^2(t) = \begin{bmatrix} -2e^{-4t} \\ 5e^{-4t} \end{bmatrix} \quad ,$$

which we can write more simply as

$$\mathbf{x}^1(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} \qquad \textit{and} \qquad \mathbf{x}^2(t) = \begin{bmatrix} -2 \\ 5 \end{bmatrix} e^{-4t} \quad .$$

37.2 More Terminology and Some Basic Results

Let us now focus on determining the basic nature of the general solutions to a homogeneous $N \times N$ linear system $\mathbf{x}' = \mathbf{P}\mathbf{x}$ where, for some positive integer N ,

$$\mathbf{P} = \mathbf{P}(t) = \begin{bmatrix} p_{11}(t) & p_{12}(t) & \cdots & p_{1N}(t) \\ p_{21}(t) & p_{22}(t) & \cdots & p_{2N}(t) \\ \vdots & \vdots & \ddots & \vdots \\ p_{N1}(t) & p_{N2}(t) & \cdots & p_{NN}(t) \end{bmatrix} \tag{37.3}$$

is a continuous $N \times N$ matrix-valued function on some interval (α, β) . Our goal is to extend the basic notions and results developed back in chapters 13 and 14 of the text for single N^{th}-order homogeneous linear equations.

The main results of our development are summarized in theorem 37.9 on page 37–12. You can go ahead and look at this theorem. Even though some of the terminology has not yet been formally defined in the context of "systems", the terminology and theorem are so similar to that already developed in chapter 13 of the text that you will probably be able to understand the gist of the theorem. That said, if we are to intelligently use theorem 37.9, we need to fully understand and verify the claims in this theorem. We will develop that understanding and verify those claims, piece by piece, in this section. This will include developing some of the concepts and terminology used in that theorem. Many of these will be concepts and terms that you should recall from your study of linear algebra.

By the way, in the following, we will also be referring to "constants", "vectors" and, maybe, "constant vectors". Just to be clear, when we refer to something as being just a constant (not constant vector), then we mean that something is a single real number. And if we refer to something as just a vector or constant vector then that something is a column vector whose N components are constants. So "$\mathbf{a}$ is a vector" means $\mathbf{a} = [a_1, a_2, \ldots, a_N]^{\mathsf{T}}$ with each a_k being some single real number.[1]

Immediate Results on Existence and Uniqueness

Let us start with the basic theorem on the existence and uniqueness of solutions to linear systems of differential equations.

Theorem 37.1 (existence and uniqueness for linear systems)
Assume $\mathbf{P}$ is a continuous $N \times N$ matrix-valued function over the interval (α, β) and $\mathbf{g}$ is a continuous vector-valued function over (α, β) , and let t_0 and $\mathbf{a}$ be, respectively, a point in (α, β) and a constant vector. Then the initial-value problem

$$\mathbf{x}' = \mathbf{P}\mathbf{x} + \mathbf{g} \qquad \text{with} \quad \mathbf{x}(t_0) = \mathbf{a} \quad ,$$

has exactly one solution over the interval (α, β) . Moreover, this solution and its derivative are continuous over that interval.

The above theorem is actually a corollary of an earlier theorem on the existence and uniqueness of solutions to systems with sufficiently continuous component functions. The verification is left to you.

[1] It is worth recalling that the set of all such column vectors with N components (with the standard definitions of vector addition and scalar multiplication) is an N-dimensional vector space.

?►Exercise 37.1: *Verify the claims in theorem 37.1 by verifying that the component functions of the system in the theorem 37.1 satisfy the requirements given in theorem 35.2 on page 765 of the published text, and then applying theorem 35.2.*

By letting $\mathbf{g} = \mathbf{0}$ in the above theorem, we get the result we will actually use in this chapter:

Lemma 37.2 (existence and uniqueness for homogeneous linear systems)
Assume $\mathbf{P}$ *is a continuous* $N \times N$ *matrix-valued function over the interval* (α, β) *, and let* t_0 *and* $\mathbf{a}$ *be, respectively, a point in the interval* (α, β) *and a constant vector. Then the initial-value problem*

$$\mathbf{x}' = \mathbf{P}\mathbf{x} \qquad \textit{with} \quad \mathbf{x}(t_0) = \mathbf{a} \quad ,$$

has exactly one solution over the interval (α, β) *. Moreover, this solution and its derivative are continuous over that interval.*

By the way, this a good place to observe that the constant zero vector-valued function,

$$\mathbf{x}(t) = \mathbf{0} \qquad \text{for all} \quad t \quad ,$$

is always a solution to any given homogeneous linear system $\mathbf{x}' = \mathbf{P}\mathbf{x}$. Consequently, the origin $(0, 0, \ldots, 0)$ is always a critical point a critical point for the given system. This is analogous to the fact that the zero function is always a solution to any given single homogeneous linear differential equation. This will be particularly significant in a few chapters.

Linear Combinations and the Principle of Superposition

Recall that a *linear combination* of $\mathbf{x}^k$'s from any finite set

$$\left\{\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^M\right\}$$

of either vectors or vector-valued functions is any expression of the form

$$c_1\mathbf{x}^1 + c_2\mathbf{x}^2 + \cdots + c_M\mathbf{x}^M$$

where the c_k's are constants. Keep in mind that, if the $\mathbf{x}^k$'s are vector-valued functions on the interval (α, β) , then

$$\mathbf{x} = c_1\mathbf{x}^1 + c_2\mathbf{x}^2 + \cdots + c_M\mathbf{x}^M$$

means

$$\mathbf{x}(t) = c_1\mathbf{x}^1(t) + c_2\mathbf{x}^2(t) + \cdots + c_M\mathbf{x}^M(t) \qquad \text{for} \quad \alpha < t < \beta \quad .$$

Now suppose we have a linear combination $c_1\mathbf{x}^1 + c_2\mathbf{x}^2 + \cdots + c_M\mathbf{x}^M$ in which each of these $\mathbf{x}^j$'s is a solution to our linear system of differential equations; that is,

$$\frac{d\mathbf{x}^j}{dt} = \mathbf{P}\mathbf{x}^j \qquad \text{for} \quad j = 1, 2, \ldots, M \quad .$$

Because of the linearity of differentiation and matrix multiplication, we then have

$$\begin{aligned}
\frac{d}{dt}\left[c_1\mathbf{x}^1 + c_2\mathbf{x}^2 + \cdots + c_M\mathbf{x}^M\right] &= c_1\frac{d\mathbf{x}^1}{dt} + c_2\frac{d\mathbf{x}^2}{dt} + \cdots + c_M\frac{d\mathbf{x}^M}{dt} \\
&= c_1\mathbf{P}\mathbf{x}^1 + c_2\mathbf{P}\mathbf{x}^2 + \cdots + c_M\mathbf{P}\mathbf{x}^M \\
&= \mathbf{P}\left[c_1\mathbf{x}^1 + c_2\mathbf{x}^2 + \cdots + c_M\mathbf{x}^M\right] \quad .
\end{aligned}$$

Cutting out the middle yields the systems version of the superposition principle:

Lemma 37.3 (principle of superposition for systems)
If $\mathbf{x}^1$, $\mathbf{x}^2$, $\ldots$ *and* $\mathbf{x}^M$ *are all solutions to a homogeneous linear system* $\mathbf{x}' = \mathbf{P}\mathbf{x}$ *, then so is any linear combination of these* $\mathbf{x}_k$ *'s .*

Observe that, if $\{\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^M\}$ is a set of solutions to our system $\mathbf{x}' = \mathbf{P}\mathbf{x}$, and $\mathbf{x}$ is any single solution equaling some linear combination of the $\mathbf{x}^k$'s at one single point t_0 in (α, β) ,

$$\mathbf{x}(t_0) \;=\; c_1\mathbf{x}^1(t_0) \;+\; c_2\mathbf{x}^2(t_0) \;+\; \cdots \;+\; c_M\mathbf{x}^M(t_0) \quad , \tag{37.4}$$

then

$$\mathbf{x} \qquad \text{and} \qquad c_1\mathbf{x}^1 \;+\; c_2\mathbf{x}^2 \;+\; \cdots \;+\; c_M\mathbf{x}^M$$

are both solutions to $\mathbf{x}' = \mathbf{P}\mathbf{x}$ satisfying the same initial condition at t_0 . But lemma 37.2 tells us that there is only one solution to this initial-value problem. Hence, $\mathbf{x}$ and this linear combination must be the same. That is,

$$\mathbf{x}(t) \;=\; c_1\mathbf{x}^1(t) \;+\; c_2\mathbf{x}^2(t) \;+\; \cdots \;+\; c_M\mathbf{x}^M(t) \quad \text{for every value } t \text{ in } (\alpha, \beta) \quad . \tag{37.5}$$

This, along with the obvious fact that equation (37.5) implies equation (37.4), gives us our next lemma.

Lemma 37.4
Let $\left\{\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^M\right\}$ *be any set of solutions to* $\mathbf{x}' = \mathbf{P}\mathbf{x}$ *, where* $\mathbf{P}$ *is a continuous* $N \times N$ *matrix-valued function on the interval* (α, β) *. Also let* $\{c_1, c_2, \ldots, c_M\}$ *be a set of constants, and let* t_0 *be a point in the interval* (α, β) *. Then, for any solution* $\mathbf{x}$ *to* $\mathbf{x}' = \mathbf{P}\mathbf{x}$ *,*

$$\mathbf{x}(t_0) \;=\; c_1\mathbf{x}^1(t_0) \;+\; c_2\mathbf{x}^2(t_0) \;+\; \cdots \;+\; c_M\mathbf{x}^M(t_0) \quad \textit{for one value } t_0 \textit{ in } (\alpha, \beta)$$

if and only if

$$\mathbf{x}(t) \;=\; c_1\mathbf{x}^1(t) \;+\; c_2\mathbf{x}^2(t) \;+\; \cdots \;+\; c_M\mathbf{x}^M(t) \quad \textit{for every value } t \textit{ in } (\alpha, \beta) \quad .$$

An application using the above lemmas is now in order. It will give you an idea of where we are heading.

!►*Example 37.2:* *We already know from exercise 37.1 that*

$$\mathbf{x}^1(t) \;=\; \begin{bmatrix}1\\1\end{bmatrix}e^{3t} \qquad \textit{and} \qquad \mathbf{x}^2(t) \;=\; \begin{bmatrix}-2\\5\end{bmatrix}e^{-4t}$$

are both solutions (over $(-\infty, \infty)$ *) to*

$$\mathbf{x}' \;=\; \mathbf{P}\mathbf{x} \qquad \textit{with} \quad \mathbf{P} \;=\; \begin{bmatrix}1 & 2\\5 & -2\end{bmatrix} \quad .$$

The principle of superposition now assures us that, for any pair c_1 *and* c_2 *of constants, the linear combination*

$$c_1\mathbf{x}^1(t) \;+\; c_2\mathbf{x}^2(t) \;=\; c_1\begin{bmatrix}1\\1\end{bmatrix}e^{3t} \;+\; c_2\begin{bmatrix}-2\\5\end{bmatrix}e^{-4t}$$

is also a solution to our homogeneous system.

The obvious question now is whether every solution is given by a linear combination of $\mathbf{x}^1$ *and* $\mathbf{x}^2$ *. To answer that, let* $\mathbf{x}(t) = [x(t), y(t)]^{\mathsf{T}}$ *be any single solution to* $\mathbf{x}' = \mathbf{P}\mathbf{x}$ *, and consider the problem of finding constants* c_1 *and* c_2 *such that*

$$\mathbf{x}(t) = c_1\mathbf{x}^1(t) + c_2\mathbf{x}^2(t) \qquad \text{for} \quad -\infty < t < \infty \quad .$$

According to our last lemma, this problem is completely equivalent to the problem of finding constants c_1 *and* c_2 *such that*

$$\mathbf{x}(t_0) = c_1\mathbf{x}^1(t_0) + c_2\mathbf{x}^2(t_0) \qquad \text{for some } t_0 \text{ in } (-\infty, \infty) \quad .$$

Letting $t_0 = 0$ *, and using the formulas for* $\mathbf{x}^1$ *and* $\mathbf{x}^2$ *, the last equation becomes*

$$\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3\cdot 0} + c_2 \begin{bmatrix} -2 \\ 5 \end{bmatrix} e^{-4\cdot 0} \quad ,$$

which we can rewrite as pair of linear algebraic equations,

$$\begin{aligned} x(0) &= 1c_1 - 2c_2 \\ y(0) &= 1c_1 + 5c_2 \end{aligned} \quad .$$

But you can easily solve this algebraic system and verify that, for each choice of $x(0)$ *and* $y(0)$ *, the one and only one solution* (c_1, c_2) *to this system is given by*

$$c_1 = \frac{1}{7}[y(0) - x(0)] \qquad \text{and} \qquad c_2 = \frac{1}{7}[6x(0) + y(0)] \quad .$$

Thus, using these values for c_1 *and* c_2 *, we have*

$$\mathbf{x}(t) = c_1\mathbf{x}^1(t) + c_2\mathbf{x}^2(t) \qquad \text{for} \quad -\infty < t < \infty \quad .$$

So, at least for the system of differential equations being considered here, the answer to the question of whether every solution is given by a linear combination of $\mathbf{x}^1$ *and* $\mathbf{x}^2$ *is yes. The above shows that, given any solution* $\mathbf{x}$ *, we can find one (and only one) corresponding pair of constants* (c_1, c_2) *such that*

$$\mathbf{x}(t) = c_1\mathbf{x}^1(t) + c_2\mathbf{x}^2(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} -2 \\ 5 \end{bmatrix} e^{-4t} \quad .$$

In other words, the above expression is a general solution to our system $\mathbf{x}' = \mathbf{P}\mathbf{x}$ *.*

As suggested in the above example, our goal is to show that, for any given $\mathbf{P}$, every solution $\mathbf{x}$ to $\mathbf{x}' = \mathbf{P}\mathbf{x}$ can be written as a linear combination of solutions from some "fundamental set" of solutions,

$$\left\{ \mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^M \right\} \quad .$$

Moreover, as illustrated in the above example, we can use lemma 37.4 us to convert the problem of finding that linear combination

$$\mathbf{x}(t) = c_1\mathbf{x}^1(t) + c_2\mathbf{x}^2(t) + \cdots + c_M\mathbf{x}^M(t) \qquad \text{for} \quad \alpha < t < \beta$$

to the problem of finding constants c_1 , c_2 , ... and c_M such that

$$\mathbf{x}(t_0) = c_1\mathbf{x}^1(t_0) + c_2\mathbf{x}^2(t_0) + \cdots + c_M\mathbf{x}^M(t_0) \quad .$$

for a single t_0. But remember that another lemma, lemma 37.2, assures us that there is a solution $\mathbf{x}$ to our system of differential equations satisfying $\mathbf{x}(t_0) = \mathbf{a}$ for each vector $\mathbf{a}$ and each t_0 in (α, β). Combining this fact with the results from lemma 37.4 gives our next lemma.

Lemma 37.5

Assume $\mathbf{P}$ be a continuous $N \times N$ matrix-valued function on the interval (α, β). Let t_0 be a point in this interval, and let

$$\left\{ \mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^M \right\}$$

be any set of solutions to $\mathbf{x}' = \mathbf{P}\mathbf{x}$. Then every solution $\mathbf{x}$ to $\mathbf{x}' = \mathbf{P}\mathbf{x}$ can be written as a linear combination of the $\mathbf{x}^k$'s if and only if every vector $\mathbf{a}$ can be written as a linear combination of vectors from the set

$$\left\{ \mathbf{x}^1(t_0), \mathbf{x}^2(t_0), \ldots, \mathbf{x}^M(t_0) \right\} \quad .$$

This lemma, along with a similar lemma concerning “linear independence”, will play a major role in our final derivations. So let’s now bring back the basic notion of linear (in)dependence.

Linear Independence

Let

$$\left\{ \mathbf{x}^1 , \mathbf{x}^2 , \ldots , \mathbf{x}^M \right\}$$

be either a set of vectors or a set of vector-valued functions on (α, β). We say that this set is *linearly independent* if and only if none these $\mathbf{x}^k$'s can be written as as linear combination of the other $\mathbf{x}^k$'s. Otherwise, we say this set is *linearly dependent*; that is, the set is linearly dependent if and only if at least one these $\mathbf{x}^k$'s can be written as as linear combination of the other $\mathbf{x}^k$'s.

Two quick observations:

1. Any constant multiple of a single $\mathbf{x}^k$ is a (very simple) linear combination of that $\mathbf{x}^k$. In particular, since $\mathbf{0} = 0\mathbf{x}^k$, any set containing the zero vector or the zero vector-valued function is automatically linearly dependent.
2. If we just have a pair $\left\{\mathbf{x}^1, \mathbf{x}^2\right\}$, the concept of linear independence simplifies to the pair being linearly independent if and only if neither $\mathbf{x}^1$ nor $\mathbf{x}^2$ is a constant multiple of the other.

!►*Example 37.3:* *Consider the set $\left\{\mathbf{x}^1, \mathbf{x}^2\right\}$ of vector-valued functions from the last example, where*

$$\mathbf{x}^1(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} \qquad \text{and} \qquad \mathbf{x}^2(t) = \begin{bmatrix} -2 \\ 5 \end{bmatrix} e^{-4t} \quad .$$

Clearly, there is no constant C such that either

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} = C \begin{bmatrix} -2 \\ 5 \end{bmatrix} e^{-4t} \qquad \text{for} \quad -\infty < t < \infty$$

or

$$\begin{bmatrix} -2 \\ 5 \end{bmatrix} e^{-4t} = C \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} \qquad \text{for} \quad -\infty < t < \infty \quad .$$

So this set of two vector-valued functions is linearly independent.

Similarly, consider the set of vectors $\left\{\mathbf{b}^1, \mathbf{b}^2\right\}$ given by the above vector-valued functions at $t = 0$,

$$\mathbf{b}^1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3 \cdot 0} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{b}^2 = \begin{bmatrix} -2 \\ 5 \end{bmatrix} e^{-4 \cdot 0} = \begin{bmatrix} -2 \\ 5 \end{bmatrix} \quad .$$

Again, it should be clear that there is no constant C such that either

$$\begin{bmatrix}1\\1\end{bmatrix} = C\begin{bmatrix}-2\\5\end{bmatrix} \qquad \text{or} \qquad \begin{bmatrix}-2\\5\end{bmatrix} = C\begin{bmatrix}1\\1\end{bmatrix} \quad .$$

So this set of two vectors is linearly independent.

Now suppose

$$\left\{ \mathbf{x}^1 , \mathbf{x}^2 , \ldots , \mathbf{x}^M \right\}$$

is a set of solutions over (α, β) to our system $\mathbf{x}' = \mathbf{P}\mathbf{x}$, and let t_0 be in (α, β). Lemma 37.4 tells us that any one solution $\mathbf{x}^j$ is a linear combination of the other $\mathbf{x}^k$'s if and only if the corresponding vector $\mathbf{x}^j(t_0)$ is a linear combination of the other $\mathbf{x}^k(t_0)$'s. This observation is worth writing down as a lemma in terms of linear independence.

Lemma 37.6
Assume $\mathbf{P}$ is a continuous $N \times N$ matrix-valued function on the interval (α, β). Let t_0 be a point in this interval, and let

$$\left\{ \mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^M \right\}$$

be any set of M solutions to $\mathbf{x}' = \mathbf{P}\mathbf{x}$. Then this set is a linearly independent set of vector-valued functions if and only if

$$\left\{ \mathbf{x}^1(t_0) , \mathbf{x}^2(t_0) , \ldots , \mathbf{x}^M(t_0) \right\}$$

is a linearly independent set of vectors.

Compare the above lemma with lemma 37.5. Both will play a major role in the following.

It will also be helpful to recall a test for linear independence that you should recall from your study of linear algebra.[2]

Lemma 37.7 (a basic test for linear independence)
A set $\left\{\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^M\right\}$ of vectors or vector-valued functions is linearly independent if and only if the only choice of constants c_1, c_2, … and c_M such that

$$c_1\mathbf{x}^1 + c_2\mathbf{x}^2 + \cdots + c_M\mathbf{x}^M = \mathbf{0}$$

is

$$c_1 = c_2 = \cdots = c_M = 0 \quad .$$

Observe that if we have two linear combinations of the same $\mathbf{x}^k$'s equaling the same $\mathbf{a}$,

$$\mathbf{a} = c_1\mathbf{x}^1 + c_2\mathbf{x}^2 + \cdots + c_M\mathbf{x}^M \qquad \text{and} \qquad \mathbf{a} = C_1\mathbf{x}^1 + C_2\mathbf{x}^2 + \cdots + C_M\mathbf{x}^M \quad ,$$

then

$$(c_1 - C_1)\mathbf{x}^1 + (c_2 - C_2)\mathbf{x}^2 + \cdots + (c_M - CM)\mathbf{x}^M = \mathbf{a} - \mathbf{a} = \mathbf{0} \quad .$$

From this, you should have no problem in verifying that the above test for linear independence is equivalent to the following "test".

[2] If you don't recall this test, see exercise 37.5 at the end of the chapter.

Lemma 37.8 (alternative test for linear independence)
Let $\{\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^M\}$ *be a set of vectors or vector-valued functions. This set is linearly independent if and only if, for each* $\mathbf{a}$ *that can be written as a linear combination of the* $\mathbf{x}^k$*'s, there is only one choice of constants* c_1 , c_2 , $\ldots$ *and* c_M *such that*

$$\mathbf{a} = c_1\mathbf{x}^1 + c_2\mathbf{x}^2 + \cdots + c_M\mathbf{x}^N \quad .$$

Fundamental Sets of Solutions

Basic Definition

We now define a *fundamental set of solutions* for $\mathbf{x}' = \mathbf{P}\mathbf{x}$ to be any linearly independent set of solutions to $\mathbf{x}' = \mathbf{P}\mathbf{x}$

$$\left\{ \mathbf{x}^1 , \mathbf{x}^2 , \ldots , \mathbf{x}^M \right\}$$

such that every solution to $\mathbf{x}' = \mathbf{P}\mathbf{x}$ can be written as a linear combination of the $\mathbf{x}^j$'s in this set.

Note that, if the above is a fundamental set of solutions for $\mathbf{x}' = \mathbf{P}\mathbf{x}$, then

$$\mathbf{x} = c_1\mathbf{x}^1 + c_2\mathbf{x}^2 + \cdots + c_N\mathbf{x}^M$$

(with the c_k's being arbitrary constants) is a general solution for $\mathbf{x}' = \mathbf{P}\mathbf{x}$.

Describing Fundamental Sets of Solutions

It turns out that there are several other ways to describe fundamental sets. To see this, let

$$\mathcal{X} = \left\{ \mathbf{x}^1 , \mathbf{x}^2 , \ldots , \mathbf{x}^M \right\}$$

be a set of solutions to $\mathbf{x}' = \mathbf{P}\mathbf{x}$ where, as usual, $\mathbf{P}$ is a continuous $N \times N$ matrix-valued function on an interval (α, β) . Take any point t_0 in the interval, and let

$$\mathcal{B} = \left\{ \mathbf{b}^1 , \mathbf{b}^2 , \ldots , \mathbf{b}^M \right\}$$

be the set of vectors given by

$$\mathbf{b}^k = \mathbf{x}^k(t_0) \qquad \text{for} \quad k = 1, 2, \ldots, M \quad .$$

From our basic definition of a 'fundamental set of solutions' we know:

> *The set* $\mathcal{X}$ *is a fundamental set of solutions to* $\mathbf{x}' = \mathbf{P}\mathbf{x}$ *if and only if* $\mathcal{X}$ *is a linearly independent set of vector-valued functions such that any solution to* $\mathbf{x}' = \mathbf{P}\mathbf{x}$ *can be written as a linear combination of the* $\mathbf{x}^k$*'s.*

From lemmas 37.5 and 37.6, we know this last statement is completely equivalent to:

> *The set* $\mathcal{X}$ *is a fundamental set of solutions to* $\mathbf{x}' = \mathbf{P}\mathbf{x}$ *if and only if* $\mathcal{B}$ *is a linearly independent set of vectors such that any vector can be written as a linear combination of the* $\mathbf{b}^k$*'s.*

Throwing in lemma 37.8 we get another equivalent statement:

> *The set* $\mathcal{X}$ *is a fundamental set of solutions to* $\mathbf{x}' = \mathbf{P}\mathbf{x}$ *if and only if, for each vector* $\mathbf{a} = [a_1, a_2, \ldots, a_N]^\mathsf{T}$ *, there is one and only one choice of constants* c_1 , c_2 , $\ldots$ *and* c_M *such that*
>
> $$\mathbf{a} = c_1\mathbf{b}^1 + c_2\mathbf{b}^2 + \cdots + c_M\mathbf{b}^M \quad .$$

At this point, you probably realize that the last two statements are saying that the set of $\mathbf{x}^k$'s is a fundamental set of solutions if and only if the set of $\mathbf{b}^k$'s is a 'basis' for the vector space of all column vectors with N components, and, from linear algebra, we know that M, the number of vectors in the set $\mathcal{B}$ must equal N the number of components in each column vector. Moreover, from linear algebra, we know that any set of N linearly independent vectors will be a basis for this space of column vectors.[3] So either of the last two statements about $\mathcal{X}$ can be rephrased as

> *The set $\mathcal{X}$ is a fundamental set of solutions to $\mathbf{x}' = \mathbf{P}\mathbf{x}$ if and only if $M = N$ and $\mathcal{B}$ is a linearly independent set of vectors.*

Applying lemma 37.6 once again with the last yields:

> *The set $\mathcal{X}$ is a fundamental set of solutions to $\mathbf{x}' = \mathbf{P}\mathbf{x}$ if and only if $M = N$ and $\mathcal{X}$ is linearly independent.*

All of the above could be considered pieces of one big lemma. Rather than state that lemma here, we will summarize the most relevant pieces in a major theorem in the next section, after making a final observation on the existence of fundamental solution sets.

Existence of Fundamental Sets of Solutions

Let us observe that fundamental sets of solutions clearly do exist. After all, no matter what N is, we can always find a linearly independent set of N vectors with N components,

$$\left\{ \mathbf{b}^1 , \mathbf{b}^2 , \ldots , \mathbf{b}^N \right\} \quad .$$

For example, if $N = 3$ we can use

$$\mathbf{b}^1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad , \quad \mathbf{b}^2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{b}^3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad ,$$

if $N = 4$ we can use

$$\mathbf{b}^1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad , \quad \mathbf{b}^2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad , \quad \mathbf{b}^3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{b}^4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad ,$$

and so on.

Lemma 37.2 then assures us that, for any point t_0 in (α, β) and every $\mathbf{b}^k$, there is a solution $\mathbf{x}^k$ to $\mathbf{x}' = \mathbf{P}\mathbf{x}$ satisfying $\mathbf{x}^k(t_0) = \mathbf{b}^k$. As noted in the last subsection above, it then follows that

$$\left\{ \mathbf{x}^1 , \mathbf{x}^2 , \ldots , \mathbf{x}^N \right\}$$

is a fundamental set of solutions to our system of differential equations.

[3] An alternative derivation not using 'basis' of the fact that $M = N$ is given in section 37.4.

37.3 The Main Result on General Solutions to Linear Systems

Looking back over the discussion on fundamental sets of solutions in the last section, you will see that we have verified the following major theorem on general solutions to linear systems of differential equations.

Theorem 37.9 (general solutions to homogenous systems)
Let $\mathbf{P}$ be a continuous $N \times N$ matrix-valued function on an interval (α, β), and consider the system of differential equations $\mathbf{x}' = \mathbf{P}\mathbf{x}$. Then all the following statements hold:

1. *Fundamental sets of solutions over (α, β) for this system exist.*
2. *Every fundamental set of solutions consists of exactly N solutions.*
3. *If $\left\{\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^N\right\}$ is any linearly independent set of N solutions to $\mathbf{x}' = \mathbf{P}\mathbf{x}$ on (α, β), then*
   (a) *$\{\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^N\}$ is a fundamental set of solutions for $\mathbf{x}' = \mathbf{P}\mathbf{x}$ on (α, β).*
   (b) *A general solution to $\mathbf{x}' = \mathbf{P}\mathbf{x}$ on (α, β) is given by*
   $$\mathbf{x}(t) = c_1\mathbf{x}^1(t) + c_2\mathbf{x}^2(t) + \cdots + c_N\mathbf{x}^N(t)$$
   where c_1, c_2, ... and c_N are arbitrary constants.
   (c) *Given any single point t_0 in (α, β) and any constant vector $\mathbf{a}$, there is exactly one ordered set of constants $\{c_1, c_2, \ldots, c_N\}$ such that*
   $$\mathbf{x}(t) = c_1\mathbf{x}^1(t) + c_2\mathbf{x}^2(t) + \cdots + c_N\mathbf{x}^N(t)$$
   satisfies the initial condition $\mathbf{x}(t_0) = \mathbf{a}$.

This theorem is the systems analog of theorem 13.5 on page 272 concerning general solutions to single N^{th}-order homogeneous linear differential equations. In fact, theorem 13.5 can be considered a corollary to the above. We verify that in section 37.6.

37.4 Wronskians and Identifying Fundamental Sets

As illustrated in the previous examples, determining whether a set of solutions is a fundamental set for our problem $\mathbf{x}' = \mathbf{P}\mathbf{x}$ is fairly easy when $\mathbf{P}$ is 2×2. Our goal now is to come up with a method for identifying a fundamental set of solutions that be easily applied when $\mathbf{P}$ is $N \times N$ even when $N > 2$.

Let us start by assuming we have a set

$$\left\{\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^M\right\}$$

of vector-valued functions on the interval (α, β), each with N components,

$$\mathbf{x}^1(t) = \begin{bmatrix} x_1^1(t) \\ x_2^1(t) \\ \vdots \\ x_N^1(t) \end{bmatrix} \quad , \quad \mathbf{x}^2(t) = \begin{bmatrix} x_1^2(t) \\ x_2^2(t) \\ \vdots \\ x_N^2(t) \end{bmatrix} \quad , \quad \dots \quad \text{and} \quad \mathbf{x}^M(t) = \begin{bmatrix} x_1^M(t) \\ x_2^M(t) \\ \vdots \\ x_N^M(t) \end{bmatrix} \quad .$$

For the moment, we need not assume the $\mathbf{x}^k$'s are solutions to our $N \times N$ system of differential equations, nor will we assume $N = M$.

A Matrix/Vector Formula for Linear Combinations

Observe:

$$c_1\mathbf{x}^1 + c_2\mathbf{x}^2 + \cdots + c_M\mathbf{x}^M = c_1 \begin{bmatrix} x_1^1 \\ x_2^1 \\ \vdots \\ x_N^1 \end{bmatrix} + c_2 \begin{bmatrix} x_1^2 \\ x_2^2 \\ \vdots \\ x_N^2 \end{bmatrix} + \cdots + c_M \begin{bmatrix} x_1^M \\ x_2^M \\ \vdots \\ x_N^M \end{bmatrix}$$

$$= \begin{bmatrix} x_1^1 c_1 + x_1^2 c_2 + \cdots + x_1^M c_M \\ x_2^1 c_1 + x_2^2 c_2 + \cdots + x_2^M c_M \\ \vdots \\ x_N^1 c_1 + x_N^2 c_2 + \cdots + x_N^M c_M \end{bmatrix}$$

$$= \begin{bmatrix} x_1^1 & x_1^2 & \cdots & x_1^M \\ x_2^1 & x_2^2 & \cdots & x_2^M \\ \vdots & \vdots & \ddots & \vdots \\ x_N^1 & x_N^2 & \cdots & x_N^M \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_M \end{bmatrix} \quad .$$

That is, for $\alpha < t < \beta$,

$$c_1\mathbf{x}^1(t) + c_2\mathbf{x}^2(t) + \cdots + c_M\mathbf{x}^M(t) = [\mathbf{X}(t)]\mathbf{c}$$

where

$$\mathbf{X}(t) = \begin{bmatrix} x_1^1(t) & x_1^2(t) & \cdots & x_1^M(t) \\ x_2^1(t) & x_2^2(t) & \cdots & x_2^M(t) \\ \vdots & \vdots & \ddots & \vdots \\ x_N^1(t) & x_N^2(t) & \cdots & x_N^M(t) \end{bmatrix} \quad \text{and} \quad \mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_M \end{bmatrix} \quad .$$

The above $N \times M$ matrix-valued function $\mathbf{X}$ will be important to us. In general, we'll simply call it the *matrix whose k^{th} column is given by* $\mathbf{x}^k$.

!►Example 37.4: *The matrix whose k^{th} column is given by $\mathbf{x}^k$ when*

$$\mathbf{x}^1(t) = \begin{bmatrix} e^{3t} \\ e^{3t} \end{bmatrix} \quad \textit{and} \quad \mathbf{x}^2(t) = \begin{bmatrix} -2e^{-4t} \\ 5e^{-4t} \end{bmatrix}$$

is

$$\mathbf{X}(t) = \begin{bmatrix} e^{3t} & -2e^{-4t} \\ e^{3t} & 5e^{-4t} \end{bmatrix} \quad .$$

Observe that, indeed,

$$[\mathbf{X}(t)]\mathbf{c} = \begin{bmatrix} e^{3t} & -2e^{-4t} \\ e^{3t} & 5e^{-4t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} c_1 e^{3t} + c_2(-2)e^{-4t} \\ c_1 e^{3t} + 5c_2 e^{-4t} \end{bmatrix}$$

$$= c_1 \begin{bmatrix} e^{3t} \\ e^{3t} \end{bmatrix} + c_2 \begin{bmatrix} -2e^{-4t} \\ e^{-4t} \end{bmatrix} = c_1\mathbf{x}^1(t) + c_2\mathbf{x}^2(t) \quad .$$

Deriving that Simple Test

Now assume these $\mathbf{x}^k$'s are solutions to our system $\mathbf{x}' = \mathbf{P}\mathbf{x}$, and let t_0 be any single value in (α, β). From lemmas 37.5, 37.6 and 37.8, we know (as noted on page 37–10 using slightly different notation) that:

> *The set* $\{\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^M\}$ *is a fundamental set of solutions to* $\mathbf{x}' = \mathbf{P}\mathbf{x}$ *if and only if, for each vector* $\mathbf{a} = [a_1, a_2, \ldots, a_N]^\mathsf{T}$ *, there is one and only one choice of constants* c_1 *,* c_2 *, ... and* c_M *such that*
>
> $$c_1\mathbf{x}^1(t_0) + c_2\mathbf{x}^2(t_0) + \cdots + c_M\mathbf{x}^M(t_0) = \mathbf{a} \quad . \tag{37.6}$$

However, from the observations made just before our last example, we know that equation (37.6) is equivalent to the algebraic system of N equations and M unknowns

$$\begin{aligned} x_1^1(t_0)c_1 + x_1^2(t_0)c_2 + \cdots + x_1^M(t_0)c_M &= a_1 \\ x_2^1(t_0)c_1 + x_2^2(t_0)c_2 + \cdots + x_2^M(t_0)c_M &= a_2 \\ &\vdots \\ x_N^1(t_0)c_1 + x_N^2(t_0)c_2 + \cdots + x_N^M(t_0)c_M &= a_N \end{aligned} \quad , \tag{37.7}$$

which can also be written as the matrix/vector equation

$$[\mathbf{X}(t_0)]\,\mathbf{c} = \mathbf{a} \tag{37.8}$$

where $\mathbf{c} = [c_1, c_2, \ldots, c_M]^\mathsf{T}$ and $\mathbf{X}(t)$ is the $N \times M$ matrix whose k^{th} column is given by $\mathbf{x}^k(t)$.

But solving either algebraic system (37.7) or matrix/vector equation (37.8) is a classic problem in linear algebra, and from linear algebra we know there is one and only one solution $\mathbf{c}$ for each $\mathbf{a}$ if and only if

$$M = N \quad \textit{and} \quad \mathbf{X}(t_0) \textit{ is invertible} \quad .$$

If these two conditions are both satisfied, then $\mathbf{c}$ can be determined from each $\mathbf{a}$ by

$$\mathbf{c} = [\mathbf{X}(t_0)]^{-1}\mathbf{a}$$

where $[\mathbf{X}(t_0)]^{-1}$ is the inverse of matrix $\mathbf{X}(t_0)$. (In practice, though, a "row reduction" method may be a more efficient way to find $\mathbf{c}$.)

Now, to make life even easier, recall that there is a relatively simple test for determining if a given square matrix $\mathbf{M}$ is invertible[4] based on the matrix's determinant, $\det(\mathbf{M})$; namely,

$$\mathbf{M} \text{ is invertible} \quad \Longleftrightarrow \quad \det(\mathbf{M}) \neq 0 \quad .$$

Thus, *our set of M solutions is a fundamental set of solutions if and only if*

$$M = N \quad \textit{and} \quad \det(\mathbf{X}(t_0)) \neq 0 \quad .$$

Wronskians and Identifying Fundamental Sets

The last line above gives us a useful test for determining if a given set of solutions is a fundamental set of solutions. It also gives the author an excuse for introducing additional terminology concerning any set

$$\left\{ \mathbf{x}^1 ,\ \mathbf{x}^2 ,\ \dots ,\ \mathbf{x}^N \right\}$$

of N vector-valued functions on an interval (α, β), with each $\mathbf{x}^k$ having N components. The *Wronskian*, W, of this set is the function on (α, β) given by

$$W(t) \; = \; \det(\mathbf{X}(t))$$

where $\mathbf{X}$ is the matrix whose k^{th} column is given by $\mathbf{x}^k$.

Using the 'Wronskian', we can now properly state the test we have just derived above.

Theorem 37.10 (Identifying Fundamental Sets of Solutions)
Let $\left\{ \mathbf{x}^1, \mathbf{x}^2, \dots, \mathbf{x}^M \right\}$ be a set of M solutions to $\mathbf{x}' = \mathbf{P}\mathbf{x}$, with $\mathbf{P}$ being a continuous $N \times N$ matrix-valued function on an interval (α, β). Then this set is a fundamental set of solutions for $\mathbf{x}' = \mathbf{P}\mathbf{x}$ if and only if both of the following hold:

1. $M = N$.
2. *For any single t_0 in (α, β), $W(t_0) \neq 0$, where W is the Wronskian of $\left\{ \mathbf{x}^1, \mathbf{x}^2, \dots, \mathbf{x}^M \right\}$.*

!►*Example 37.5:* *It is not hard to verify that three solutions (on $(-\infty, \infty)$ to*

$$\mathbf{x}' \; = \; \mathbf{P}\mathbf{x} \qquad \textit{with} \quad \mathbf{P} \; = \; \begin{bmatrix} 2 & -\frac{12}{5} & \frac{4}{5} \\ 2 & -3 & 1 \\ 0 & \frac{6}{5} & \frac{8}{5} \end{bmatrix}$$

are

$$\mathbf{x}^1 \; = \; \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} e^{2t} \quad , \quad \mathbf{x}^2 \; = \; \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} e^{-2t} \quad \textit{and} \quad \mathbf{x}^3 \; = \; \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix} e^{2t} \quad .$$

---

[4] More terminology you should recall:

$$\mathbf{M} \text{ is singular} \quad \Longleftrightarrow \quad \mathbf{M} \text{ is not invertible}$$

$$\mathbf{M} \text{ is nonsingular} \quad \Longleftrightarrow \quad \mathbf{M} \text{ is invertible} \quad .$$

The corresponding matrix whose k^{th} columm given by $\mathbf{x}^k$ is

$$\mathbf{X}(t) = \begin{bmatrix} e^{2t} & 2e^{-2t} & 3e^{2t} \\ e^{2t} & 3e^{-2t} & e^{2t} \\ 3e^{2t} & -e^{-2t} & 3e^{2t} \end{bmatrix}$$

and the Wronskian is

$$W(t) = \det(\mathbf{X}(t)) = \begin{vmatrix} e^{2t} & 2e^{-2t} & 3e^{2t} \\ e^{2t} & 3e^{-2t} & e^{2t} \\ 3e^{2t} & -e^{-2t} & 3e^{2t} \end{vmatrix} \quad .$$

Computing out this determinant is not difficult, but not necessary. All we need is to compute is $W(t_0)$ for some convenient value t_0, say $t_0 = 0$,

$$\begin{aligned} W(0) = \det(\mathbf{X}(0)) &= \begin{vmatrix} e^{2\cdot 0} & 2e^{-2\cdot 0} & 3e^{2\cdot 0} \\ e^{2\cdot 0} & 3e^{-2\cdot 0} & e^{2\cdot 0} \\ 3e^{2\cdot 0} & -e^{-2\cdot 0} & 3e^{2\cdot 0} \end{vmatrix} \\ &= \begin{vmatrix} 1 & 2 & 3 \\ 1 & 3 & 1 \\ 3 & -1 & 3 \end{vmatrix} \\ &= 1\begin{vmatrix} 3 & 1 \\ -1 & 3 \end{vmatrix} - 2\begin{vmatrix} 1 & 1 \\ 3 & 3 \end{vmatrix} + 3\begin{vmatrix} 1 & 3 \\ 3 & -1 \end{vmatrix} \\ &= 1[9+1] - 2[3-3] + 3[-1-9] = -20 \quad . \end{aligned}$$

Since $W(0) \neq 0$, the above theorem tells us that the set $\{\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3\}$ is a fundamental set of solutions for the above system of differential equations. And that means

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} e^{-2t} + c_3 \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix} e^{2t}$$

is a general solution to the 3×3 system $\mathbf{x}' = \mathbf{P}\mathbf{x}$ being considered here.

By the way, the fact that we can choose t_0 arbitrarily in (α, β) tells us that whether $W(t_0)$ is zero or not is totally independent of the choice of t_0. That gives us the following corollary.

Corollary 37.11

Assume $\mathbf{P}$ is a continuous $N \times N$ matrix-valued function on an interval (α, β), and let W be the Wronskian of a set of N solutions to $\mathbf{x}' = \mathbf{P}\mathbf{x}$. Then

$$W(t_0) \neq 0 \quad \text{for one value } t_0 \text{ in } (\alpha, \beta)$$

if and only if

$$W(t) \neq 0 \quad \text{for every value } t \text{ in } (\alpha, \beta) \quad .$$

37.5 Fundamental Matrices

In the last section, we introduced the matrix-valued function $\mathbf{X}$ whose k^{th} column is given by the k^{th} vector-valued function in a set

$$\left\{\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^N\right\} \quad .$$

In the future, we will refer to $\mathbf{X}$ as a *fundamental matrix* for $\mathbf{x}' = \mathbf{P}\mathbf{x}$ if and only if the above set is a fundamental set of solutions for $\mathbf{x}' = \mathbf{P}\mathbf{x}$. Fundamental matrices will play a role in some of our later discussions.

!►*Example 37.6:* *In example 37.5, just above, we considered the problem*

$$\mathbf{x}' = \mathbf{P}\mathbf{x} \qquad \textit{with} \quad \mathbf{P} = \begin{bmatrix} 2 & -\frac{12}{5} & \frac{4}{5} \\ 2 & -3 & 1 \\ 0 & \frac{6}{5} & \frac{8}{5} \end{bmatrix} \quad ,$$

and saw that the set $\{\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3\}$ *with*

$$\mathbf{x}^1 = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} e^{2t} \quad , \quad \mathbf{x}^2 = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} e^{-2t} \quad \textit{and} \quad \mathbf{x}^3 = \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix} e^{2t}$$

is a fundamental set of solutions to the given problem $\mathbf{x}' = \mathbf{P}\mathbf{x}$. *Hence, the matrix whose* k^{th} *columm given by* $\mathbf{x}^k$,

$$\mathbf{X}(t) = \begin{bmatrix} e^{2t} & 2e^{-2t} & 3e^{2t} \\ e^{2t} & 3e^{-2t} & e^{2t} \\ 3e^{2t} & -e^{-2t} & 3e^{2t} \end{bmatrix} \quad ,$$

is a fundamental matrix for this problem.

For future reference, let us note that the following lemma follows immediately from the discussion in the previous section:

Lemma 37.12
If $\mathbf{X}$ *is a fundamental matrix for* $\mathbf{x}' = \mathbf{P}\mathbf{x}$ *where* $\mathbf{P}$ *is a continuous matrix-valued function on an interval* (α, β), *then* $\mathbf{X}(t)$ *is an invertible matrix for each* t *in* (α, β).

37.6 General Solutions for a Single N^{th}-order Linear Differential Equation*

In chapter 13 we presented two important theorems concerning single N^{th}-order linear differential equations — the main theorem on general solutions, theorem 13.5 on page 272, and a theorem

* The material in this section plays no role in later developments, and can be safely skipped by those more interested in learning more about systems of differential equations than in verifying theorems for single equations.

on using Wronskians to identify fundamental sets of solutions, theorem 13.6 on page 274. Those theorems were proven in that and the following chapter for the case where $N = 2$, but were not completely verified assuming $N > 2$. Using the results just derived in the chapter, we can now do so. Let us see how.

The Basic Assumptions

Throughout this section, we are concerned with a fairly general N^{th}-order linear differential equation over an interval (α, β)

$$a_0 y^{(N)} + a_1 y^{(N-1)} + \cdots + a_{N-2} y'' + a_{N-1} y' + a_N y = 0 \tag{37.9a}$$

where each a_k is a continuous function over (α, β) and with $a_0(t)$ never being zero in that interval. Solving for $y^{(N)}$, we see that this equation can also be written as

$$y^{(N)} = p_{N1} y + p_{N2} y' + \cdots + p_{NN} y^{(N-1)} \tag{37.9b}$$

where

$$p_{N1} = -\frac{a_N}{a_0} \quad , \quad p_{N2} = -\frac{a_{N-1}}{a_0} \quad , \quad \ldots \quad \text{and} \quad p_{NN} = -\frac{a_1}{a_0} \quad .$$

Observe that the assumptions made on the a_k's ensure that each of these p_{Nk}'s is a continuous function on (α, β).

On occasion, we will also be interested in a solution to the above differential equation that satisfies some N^{th}-order set of initial conditions

$$\begin{gathered} y(t_0) = a_1 \quad , \quad y'(t_0) = a_2 \quad , \\ y''(t_0) = a_3 \quad , \quad \cdots \quad \text{and} \quad y^{(N-1)}(t_0) = a_N \quad . \end{gathered} \tag{37.10}$$

where t_0 is some point in (α, β) and the A_k's are real numbers.

The Analysis

As discussed in section 35.6 of the published text, we can convert equation (37.9b) to an equivalent standard $N \times N$ system $\mathbf{x}' = \mathbf{F}(t, \mathbf{x})$ by introducing N new unknown functions x_1, x_2, ... and x_N related to y and each other by

$$\begin{gathered} x_1 = y \quad , \quad x_2 = {x_1}' = y' \\ x_3 = {x_2}' = y'' \quad , \quad \ldots \quad \text{and} \quad x_N = {x_{N-1}}' = y^{(N-1)} \quad , \end{gathered}$$

and observing that, because of differential equation (37.9b),

$$ {x_N}' = y^N = p_{N1} x_1 + p_{N2} x_2 + \cdots + p_{NN} x_N \quad .$$

This gives us the standard system

$$\begin{bmatrix} {x_1}' \\ {x_2}' \\ \vdots \\ {x_{N-1}}' \\ {x_N}' \end{bmatrix} = \begin{bmatrix} x_2 \\ x_3 \\ \vdots \\ x_N \\ p_{N1} x_1 + p_{N2} x_2 + \cdots + p_{NN} x_N \end{bmatrix} \quad ,$$

which we immediately recognize as being a homogeneous linear system $\mathbf{x}' = \mathbf{P}\mathbf{x}$ with

$$\mathbf{P} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & \\ 0 & 0 & 1 & 0 & \cdots & 0 & \\ 0 & 0 & 0 & 1 & \cdots & 0 & \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & \\ p_{N1} & p_{N2} & p_{N3} & p_{N4} & \cdots & p_{NN} & \end{bmatrix} \quad .$$

Moreover, since the p_{Nk}'s are continuous functions on (α, β), it should be clear that $\mathbf{P}$ is a continuous matrix-valued function on (α, β)

In addition, because of the relations between the x_k's and the derivatives of y, set (37.10) of initial conditions becomes the initial condition $\mathbf{x}(t_0) = \mathbf{a}$ where

$$\mathbf{a} = [a_1 , a_2 , \dots , a_N]^{\mathsf{T}} \quad .$$

It should further be clear that:

1. If y is a solution to differential equation (37.9a) which satisfies initial condition set (37.10), and

$$\mathbf{x} = \begin{bmatrix} y & y' & y'' & \cdots & y^{(N-1)} \end{bmatrix}^{\mathsf{T}} \quad ,$$

then $\mathbf{x}$ is a solution to $\mathbf{x} = \mathbf{P}\mathbf{x}$ satisfying initial condition $\mathbf{x}(t_0) = \mathbf{a}$.

2. If $\mathbf{x} = [x_1, x_2, \dots, x_N]^{\mathsf{T}}$ is a solution to $\mathbf{x} = \mathbf{P}\mathbf{x}$ which satisfies initial condition $\mathbf{x}(t_0) = \mathbf{a}$, and $y = x_1$, then

(a) $\mathbf{x} = [y, y', y'', \dots, y^{N-1)}]^{\mathsf{T}}$, and

(b) y is a solution to differential equation (37.9a) satisfying initial condition (37.10).

Thus, for each solution y to differential equation (37.10) there is a single corresponding solution $\mathbf{x}$ to $\mathbf{x}' = \mathbf{P}\mathbf{x}$. Likewise, for each solution $\mathbf{x}$ to $\mathbf{x}' = \mathbf{P}\mathbf{x}$ there is a corresponding solution y to differential equation (37.10). Moreover, these corresponding pairs are related by

$$\mathbf{x} = \begin{bmatrix} y & y' & y'' & \cdots & y^{(N-1)} \end{bmatrix}^{\mathsf{T}} \quad .$$

So now let

$$\{ y_1 , y_2 , \dots , y_M \} \qquad \text{and} \qquad \{ \mathbf{x}^1 , \mathbf{x}^2 , \dots , \mathbf{x}^M \}$$

be, respectively, a set of functions and vector-valued functions on (α, β) with each y_k and $\mathbf{x}^k$ related as just described,

$$\mathbf{x}^k = \begin{bmatrix} y & y_k' & y_k'' & \cdots & y_k{}^{(N-1)} \end{bmatrix}^{\mathsf{T}} \quad .$$

Further assume that either all the y_k's are solutions to differential equation (37.10) OR that all the $\mathbf{x}^k$'s are solutions to $\mathbf{x}' = \mathbf{P}\mathbf{x}$. Then the above assures us that, in fact, all the y_k's are solutions to differential equation (37.10) AND that all the $\mathbf{x}^k$'s are solutions to $\mathbf{x}' = \mathbf{P}\mathbf{x}$. Moreover, you should have little trouble in verifying each of the following:

1. Let y and $\mathbf{x}$ be a corresponding pair of solutions to differential equation (37.10) and $\mathbf{x}' = \mathbf{P}\mathbf{x}$ (as described above), respectively, and let c_1, c_2, ... nad c_M be some collection of constants. Then

$$y = c_1 y_1 + c_2 y_2 + \cdots + c_M y_M$$

if and only if

$$\mathbf{x} = c_1 \mathbf{x}^1 + c_2 \mathbf{x}^2 + \cdots + c_M \mathbf{x}^M \quad .$$

2. Let c_1, c_2, ... and c_M be some collection of constants, and let
$$y = c_1 y_1 + c_2 y_2 + \cdots + c_M y_M$$
and
$$\mathbf{x} = c_1 \mathbf{x}^1 + c_2 \mathbf{x}^2 + \cdots + c_M \mathbf{x}^M \quad .$$
Then y satisfies the initial condition set (37.10) if and only if $\mathbf{x}(t_0) = \mathbf{a}$.

3. Every solution of differential equation (37.10) can be written as a linear combination of the y_k's if and only if every solution of $\mathbf{x}' = \mathbf{P}\mathbf{x}$ can be written as a linear combination of the $\mathbf{x}^k$'s.

4. The set of y_k's is linearly independent if and only if the set of $\mathbf{x}^k$'s is linearly independent.

5. The set of y_k's is a fundamental set of solution to differential equation (37.10) if and only if the set of $\mathbf{x}^k$'s is a fundamental set of solutions to $\mathbf{x}' = \mathbf{P}\mathbf{x}$.

?►*Exercise 37.2:* *Justify each of the five claims made above.*

It should now be very clear why theorems 13.5 and 37.9 are so similar, and why it was claimed that theorem 13.5 (which we did not actually prove) can be derived from theorem 37.9 (which we did prove). Why don't you do that derivation yourself?

?►*Exercise 37.3:* *Using the above results and theorem 37.9, verify the claims in theorem 13.5.*

Finally, let's deal with the claims made in chapter 13 concerning Wronskians. Let
$$\{ y_1 , y_2 , \ldots , y_N \}$$
be any set of N solutions to differential equation (37.10), and let
$$\left\{ \mathbf{x}^1 , \mathbf{x}^2 , \ldots , \mathbf{x}^N \right\}$$
be the corresponding set of solutions to $\mathbf{x}' = \mathbf{P}\mathbf{x}$ with each y_k and $\mathbf{x}^k$ related by
$$\mathbf{x}^k = \begin{bmatrix} y & y_k{}' & y_k{}'' & \cdots & y_k{}^{(N-1)} \end{bmatrix}^{\mathsf{T}} \quad .$$
Starting with the definition of Wronskian given in chapter 13, we see that
$$\text{Wronskian of } \{y_1, y_2, \ldots, y_N\} = \begin{vmatrix} y_1 & y_2 & y_3 & \cdots & y_N \\ y_1{}' & y_2{}' & y_3{}' & \cdots & y_N{}' \\ y_1{}'' & y_2{}'' & y_3{}'' & \cdots & y_N{}'' \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ y_1{}^{(N-1)} & y_2{}^{(N-1)} & y_3{}^{(N-1)} & \cdots & y_N{}^{(N-1)} \end{vmatrix}$$
$$= \begin{vmatrix} x_1^1 & x_1^2 & \cdots & x_1^N \\ x_2^1 & x_2^2 & \cdots & x_2^N \\ \vdots & \vdots & \ddots & \vdots \\ x_N^1 & x_N^2 & \cdots & x_N^N \end{vmatrix}$$
$$= \text{Wronskian of } \left\{ \mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^N \right\} \quad .$$

Using this, it is a simple matter to show that the general results on Wronskians given in chapter 13 (theorem 13.6 on page 274) follow from the results concerning Wronskians proven in this chapter (theorem 37.10 and corollary 37.11).

Additional Exercises

37.4. *Rewrite each of the following linear systems of differential equations in matrix/vector form.*

a. $$\begin{aligned} x' &= 3x + 5y \\ y' &= -5x + 7y \end{aligned}$$

b. $$\begin{aligned} {x_1}' &= x_2 \\ {x_2}' &= x_3 \\ {x_3}' &= 4x_1 + 3x_2 - 2x_3 \end{aligned}$$

c. $$\begin{aligned} x' &= 2x - y + 2z + 4 \\ y' &= 2y - 4z + 5 \\ z' &= 9x - 3z + 6 \end{aligned}$$

d. $$\begin{aligned} {x_1}' &= 2x_2 - t^2 x_3 + \sin(t) \\ {x_2}' &= (t+1)x_1 + tx_3 - \cos(t) \\ {x_3}' &= 3t^3 x_2 + \sqrt{t} \end{aligned}$$

37.5. *Consider the two equations*

$$\mathbf{x}^M = C_1\mathbf{x}^1 + C_2\mathbf{x}^2 + \cdots + C_{M-1}\mathbf{x}^{M-1} \quad . \tag{37.11}$$

and

$$c_1\mathbf{x}^1 + c_2\mathbf{x}^2 + \cdots + c_M\mathbf{x}^M = \mathbf{0} \quad . \tag{37.12}$$

where $\{\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^M\}$ *is a set of vector-valued functions on an interval* (α, β) .

a. *Using simple algebra, show that equation (37.11) holds for some constants* C_1 , C_2 , $\ldots$ *and* C_{M-1} *if and only if equation (37.12) holds for some constants* c_1 , c_2 , $\ldots$ *and* c_M *with* $c_M \neq 0$.

b. *Expand on the above and explain how it follows that at least one of the* $\mathbf{x}^k$*'s must be a linear combination of the other* $\mathbf{x}^k$*'s if and only if equation (37.12) holds with at least one of the* c_k*'s being nonzero.*

c. *Finish proving lemma 37.7 on page 37–9.*

37.6. *Consider the system*

$$\begin{aligned} x' &= y \\ y' &= -4t^{-2}x + 3t^{-1}y \end{aligned} \quad .$$

a. *Rewrite this system in matrix/vector form.*

b. *What are the largest intervals over which we can be sure solutions to this system exist?*

c. *Verify that*

$$\mathbf{x}^1(t) = \begin{bmatrix} t^2 \\ 2t \end{bmatrix} \qquad \text{and} \qquad \mathbf{x}^2(t) = \begin{bmatrix} t^2 \ln|t| \\ t(1 + 2\ln|t|) \end{bmatrix}$$

are both solutions to this system.

d. *Compute the Wronskian* $W(t)$ *of the set of the above* $\mathbf{x}^k$*'s at some convenient nonzero point* $t = t_0$ *(part of this problem is to choose a convenient point). What does this value of* $W(t_0)$ *tell you?*

e. *Using the above, find the solution to the above system satisfying*

i. $\mathbf{x}(1) = [1, 0]^{\mathsf{T}}$ **ii.** $\mathbf{x}(1) = [0, 1]^{\mathsf{T}}$

37.7. *Consider the system*

$$\begin{aligned} x' &= 0x + 2y - 2z \\ y' &= -2x + 4y - 2z \\ z' &= 2x + 2y - 4z \end{aligned} \quad .$$

a. *Rewrite this system in matrix/vector form.*

b. *What is the largest interval over which we are sure solutions to this system exist?*

c. *Verify that*

$$\mathbf{x}(t) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad , \quad \mathbf{y}(t) = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} e^{-2t} \quad \text{and} \quad \mathbf{z}(t) = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} e^{2t}$$

are all solutions to this system.

d. *Compute the Wronskian* $W(t)$ *of the set* $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ *some convenient point* $t = t_0$ *(choosing a convenient point is part of the problem), and verify that the above* $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ *is a fundamental set of solutions to the above system of differential equations.*

37.8. *Four solutions to*

$$\mathbf{x}' = \begin{bmatrix} 0 & -2 & 0 \\ 1 & 0 & 1 \\ 0 & -2 & 0 \end{bmatrix} \mathbf{x}$$

are

$$\mathbf{x}^1(t) = \begin{bmatrix} \cos(2t) \\ \sin(2t) \\ \cos(2t) \end{bmatrix} \quad , \quad \mathbf{x}^2(t) = \begin{bmatrix} \sin(2t) \\ -\cos(2t) \\ \sin(2t) \end{bmatrix} \quad , \quad \mathbf{x}^3(t) = \begin{bmatrix} -\sin^2(t) \\ \sin(t)\cos(t) \\ \cos^2(t) \end{bmatrix}$$

and

$$\mathbf{x}^4(t) = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \quad .$$

Given this, determine which of the following are fundamental sets of solutions to the given system:

a. $\left\{\mathbf{x}^1, \mathbf{x}^2\right\}$ **b.** $\left\{\mathbf{x}^1, \mathbf{x}^4\right\}$ **c.** $\left\{\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3\right\}$

d. $\left\{\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^4\right\}$ **e.** $\left\{\mathbf{x}^1, \mathbf{x}^3, \mathbf{x}^4\right\}$ **f.** $\left\{\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3, \mathbf{x}^4\right\}$

37.9. *Four solutions to*

$$\mathbf{x}' = \begin{bmatrix} -1 & -1 & 2 \\ -8 & 1 & 4 \\ -4 & -1 & 5 \end{bmatrix} \mathbf{x}$$

are

$$\mathbf{x}^1(t) = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} e^{3t} \quad , \quad \mathbf{x}^2(t) = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} e^{3t} \quad , \quad \mathbf{x}^3(t) = \begin{bmatrix} 3 \\ -4 \\ 4 \end{bmatrix} e^{3t}$$

and

$$\mathbf{x}^4(t) = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} e^{-t} \quad .$$

Given this, determine which of the following are fundamental sets of solutions to the given system:

a. $\left\{\mathbf{x}^1, \mathbf{x}^2\right\}$ **b.** $\left\{\mathbf{x}^1, \mathbf{x}^4\right\}$ **c.** $\left\{\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3\right\}$

d. $\left\{\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^4\right\}$ **e.** $\left\{\mathbf{x}^1, \mathbf{x}^3, \mathbf{x}^4\right\}$ **f.** $\left\{\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3, \mathbf{x}^4\right\}$

37.10. *Traditionally (i.e., in most other texts), corollary 37.11 on page 37–16 is usually proven by showing that the Wronskian W of a set of N solutions to an $N \times N$ system $\mathbf{x}' = \mathbf{P}\mathbf{x}$ satisfies the differential equation*

$$W' = \left[p_{1,1} + p_{2,2} + \cdots + p_{N,N}\right] W \quad ,$$

and then solving this differential equation and verifying that the solution is nonzero over the interval of interest if and only if it is nonzero at one point in the interval. Do this yourself for the case where $N = 2$.

Some Answers to Some of the Exercises

WARNING! Most of the following answers were prepared hastily and late at night. They have not been properly proofread! Errors are likely!

4a. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ -5 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

4b. $\begin{bmatrix} {x_1}' \\ {x_2}' \\ {x_3}' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & 3 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

4c. $\begin{bmatrix} {x_1}' \\ {x_2}' \\ {x_3}' \end{bmatrix} = \begin{bmatrix} 2 & -1 & 2 \\ 0 & 2 & -4 \\ 9 & 0 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$

4d. $\begin{bmatrix} {x_1}' \\ {x_2}' \\ {x_3}' \end{bmatrix} = \begin{bmatrix} 0 & 2 & -t^2 \\ t+1 & 0 & t \\ 0 & 3t^3 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} \sin(t) \\ -\cos(t) \\ \sqrt{t} \end{bmatrix}$

6a. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -4t^{-2} & 3t^{-1} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

6b. $(-\infty, 0)$ and $(0, \infty)$

6d. $W(1) = 1 \neq 0$ (Hence $\{\mathbf{x}^1, \mathbf{x}^2\}$ is a fundamental set of solutions.)

6e i. $\mathbf{x}(t) = \mathbf{x}^1(t) - 2\mathbf{x}^2(t) = \begin{bmatrix} t^2(1 - 2\ln|t|) \\ -4t\ln|t| \end{bmatrix}$

6e ii. $\mathbf{x}(t) = \mathbf{x}^2(t) = \begin{bmatrix} t^2 \ln|t| \\ t(1 + 2\ln|t|) \end{bmatrix}$

7a. $\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 0 & 2 & -2 \\ -2 & 4 & -2 \\ 2 & 2 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

7b. $(-\infty, \infty)$

7d. $W(0) = -1$

8a. It is not a fundamental set since – the set is too small.

8b. It is not a fundamental set since – the set is too small.

8c. It is a fundamental set – there are three solutions in the set, and $W(0) \neq 0$.

8d. Is a fundamental set – there are three solutions in the set, and $W(0) \neq 0$.

8e. Is not a fundamental set – there are three solutions in the set, but $W(0) = 0$.

8f. Is not a fundamental set – the set is too large.

9a. It is not a fundamental set – the set is too small.

9b. It is not a fundamental set – the set is too small.

9c. It is not a fundamental set – there are three solutions in the set, but $W(0) = 0$.

9d. It is a fundamental set – there are three solutions in the set, and $W(0) \neq 0$.

9e. It is a fundamental set – there are three solutions in the set, and $W(0) \neq 0$.

9f. It is not a fundamental set – the set is too large.