Contents

V Power Series and Modified Power Series Solutions

30 Series Solutions: Preliminaries **30–1**
30.1 Infinite Series . . . 30–1
Basic Basics . . . 30–1
The Geometric Series . . . 30–3
Absolute Convergence and Convergence Tests . . . 30–5
30.2 Power Series and Analytic Functions . . . 30–6
Definition and Examples . . . 30–6
Convergence and the Radius of Convergence . . . 30–8
Algebra with Power Series and Analytic Functions . . . 30–9
Calculus with Power Series and Analytic Functions . . . 30–12
30.3 Elementary Complex Analysis . . . 30–16
The Complex Plane . . . 30–16
Power Series and Analytic Functions . . . 30–17
30.4 Additional Basic Material That May Be Useful . . . 30–19
Two More General Tests for Convergence . . . 30–19
More on Algebra with Power Series and Analytic Functions . . . 30–20
Partial Sum Approximations with Taylor Series . . . 30–23
Exercises . . . 30–24
Some Answers to Some of the Exercises . . . 30–28

31 Power Series Solutions I: Basic Computational Methods **31–1**
31.1 Basics . . . 31–1
General Power Series Solutions . . . 31–1
The Two Methods, Briefly . . . 31–2
31.2 The Algebraic Method with First-Order Equations . . . 31–3
Details of the Method . . . 31–3
Practical Advice on Using the Method . . . 31–8
Initial-Value Problems (and Finding Patterns, Again) . . . 31–13
31.3 Validity of of the Algebraic Method for First-Order Equations . . . 31–14
Non-Existence of Power Series Solutions . . . 31–14
Singular and Ordinary Points, and the Radius of Analyticity . . . 31–15
Validity of the Algebraic Method . . . 31–16
Identifying Singular and Ordinary Points . . . 31–16

31.4 The Algebraic Method with Second-Order Equations 31–19
Details of the Method 31–19
Practical Advice on Using the Method 31–24
Initial-Value Problems (and Finding Patterns) 31–24
31.5 Validity of the Algebraic Method for Second-Order Equations 31–26
Nonexistence of Power Series Solutions 31–26
Validity of the Algebraic Method 31–27
Identifying Singular and Ordinary Points 31–27
31.6 The Taylor Series Method 31–29
The Basic Idea (Expanded) 31–29
The Steps in the Taylor Series Method 31–29
Validity of the Solutions 31–34
31.7 Appendix: Using Induction 31–34
The Basic Ideas 31–34
Usage Notes 31–38
The General Principle of Induction 31–39
Exercises 31–39
Some Answers to Some of the Exercises 31–46

32 Power Series Solutions II: Generalizations and Theory 32–1
32.1 Equations with Analytic Coefficients 32–1
32.2 Ordinary and Singular Points, the Radius of Analyticity, and the Reduced Form 32–2
Introducing Complex Variables 32–2
Ordinary and Singular Points 32–4
Radius of Analyticity 32–6
32.3 The Reduced Forms 32–6
A Standard Way to Rewrite Our Equations 32–6
Ordinary Points and the Reduced Form 32–7
32.4 Existence of Power Series Solutions 32–8
Deriving the Generic Recursion Formulas 32–8
Validity of the Power Series Solutions 32–9
32.5 The Radius of Convergence for the Solution Series 32–14
What We Have, and What We Need to Show 32–14
Constructing the Series for Comparison 32–15
Convergence of the Comparison Series 32–16
32.6 Singular Points and the Radius of Convergence 32–18
32.7 Appendix: A Brief Overview of Complex Calculus 32–18
Functions of a Complex Variable 32–18
Complex Differentiability 32–19
Testing for Complex Diffentiability 32–20
Differentiability of an Analytic Function 32–21
Complex Differentiability and Analyticity 32–21
32.8 Appendix: The "Closest Singular Point" 32–23
The Problem and Fundamental Theorem 32–23
Verifying Theorem 32.20 32–24
32.9 Appendix: Singular Points and the Radius of Convergence for Power Series Solutions 32–27
Analytic Continuation 32–27

Ordinary and Singular Points for Power-Series Functions 32–29
Complex Power Series Solutions 32–32
Singular Points of Differential Equations and Solutions 32–34
Exercises 32–35
Some Answers to Some of the Exercises 32–38

33 Modified Power Series Solutions and the Basic Method of Frobenius 33–1
33.1 Euler Equations and Their Solutions 33–1
33.2 Regular and Irregular Singular Points (and the Frobenius Radius of Convergence) 33–6
Basic Terminology 33–6
33.3 The (Basic) Method of Frobenius 33–10
Motivation and Preliminary Notes 33–10
The (Basic) Method of Frobenius 33–12
“First” and “Second” Solutions 33–21
33.4 Basic Notes on Using the Frobenius Method 33–22
The Obvious 33–22
Solutions on Intervals with $x < x_0$ 33–23
Convergence of the Series 33–23
Variations of the Method 33–24
Using the Method When x_0 is an Ordinary Point 33–25
33.5 About the Indicial and Recursion Formulas 33–25
The Indicial Equation and the Exponents 33–26
The Recursion Formulas 33–27
Problems Possibly Arising in Step 8 33–28
33.6 Dealing with Complex Exponents 33–32
33.7 Appendix: On Tests for Regular Singular Points 33–33
Proof of Theorem 33.2 33–33
Testing for Regularity When the Coefficients Are Not Rational 33–35
Exercises 33–36
Some Answers to Some of the Exercises 33–40

34 The Big Theorem on the Frobenius Method, With Applications 34–1
34.1 The Big Theorems 34–1
The Theorems 34–1
Alternate Formulas 34–3
Theorem 34.2 and the Method of Frobenius 34–4
34.2 Local Behavior of Solutions: Issues 34–5
34.3 Local Behavior of Solutions: Limits at Regular Singular Points 34–6
Preliminary Approximations 34–6
Solutions Corresponding to r_1 34–7
Solutions Corresponding to r_2 when $r_2 \neq r_1$ 34–7
Solutions Corresponding to r_2 when $r_2 = r_1$ 34–8
Derivatives 34–10
34.4 Local Behavior: Analyticity and Singularities in Solutions 34–10
34.5 A Case Study: The Legendre Equations 34–12
What We Already Know 34–13
The Singular Points of the Solutions 34–13
Solution Limits at $x = 1$ 34–14

Solution Limits at $x = -1$ . . . 34–15
The Unboundedness of the Nonpolynomial Solutions . . . 34–16
The Polynomial Solutions and Legendre Polynomials . . . 34–16
Summary . . . 34–17
34.6 Finding Second Solutions Using Theorem 34.2 . . . 34–17
The Second Solution When $r_1 - r_2$ Is a Positive Integer . . . 34–18
The Second Solution When $r_1 = r_2$ . . . 34–21
Exercises . . . 34–21
Some Answers to Some of the Exercises . . . 34–26

35 Validating the Method of Frobenius **35–1**
35.1 Basic Assumptions and Symbology . . . 35–1
35.2 The Indicial Equation and Basic Recursion Formula . . . 35–2
Basic Derivations . . . 35–2
The Indicial Equation . . . 35–4
Recursion Formulas . . . 35–6
The Basic Recursion Formula . . . 35–6
35.3 The Easily Obtained Series Solutions . . . 35–7
Solutions Corresponding to r_1 . . . 35–8
"Unexceptional" Solutions Corresponding to r_2 . . . 35–8
Deriving the "Exceptional" Solutions . . . 35–9
35.4 Second Solutions When $r_2 = r_1$ . . . 35–10
35.5 Second Solutions When $r_1 - r_2 = K$ . . . 35–13
Preliminaries . . . 35–13
The Case Where We Get Lucky . . . 35–13
The Other Case . . . 35–14
Verifying Our Solution . . . 35–17
35.6 Convergence of the Solution Series . . . 35–20
Assumptions and Claim . . . 35–20
The Proof . . . 35–21

VI Systems of Differential Equations

36 Systems of Differential Equations: Basics **36–1**
36.1 General Introduction . . . 36–1
Basic Terminology and Notions . . . 36–1
36.2 A Few Illustrative Applications . . . 36–5
A Falling Object . . . 36–5
Mixing Problems with Multiple Tanks . . . 36–6
Foxes in the Rabbit Ranch (A Predator-Prey Model) . . . 36–8
Double Mass-Spring System . . . 36–9
36.3 Converting High-Order Differential Equations and Systems to Simple First-Order Systems . . . 36–11
Converting Single Differential Equations . . . 36–11
Converting Higher-Order Systems . . . 36–13
36.4 The Pendulum . . . 36–14
Exercises . . . 36–16

Some Answers to Some of the Exercises 36–20

37 Standard First-Order Systems: Basics **37–1**

37.1 "Standard" First-Order Systems 37–1

Basic Terminology 37–1

Matrix/Vector Notation for Systems 37–3

Constant (or Equilibrium) Solutions 37–6

"Graphing" 37–8

37.2 Sketching Trajectories for Autonomous Systems 37–11

The Two-Dimensional Case 37–11

Direction Fields 37–11

Higher-Order Cases 37–14

37.3 Critical Points, Stability and Long-Term Behavior 37–15

Critical Points and Stability 37–15

37.4 Existence and Uniqueness of Solutions and Trajectories 37–18

Existence and Uniqueness of Solutions 37–18

Trajectories for Regular Autonomous Systems 37–19

37.5 Proving Theorem 37.5 37–21

The Assumptions 37–21

Preliminaries 37–21

Finishing the Proof of Theorem 37.5 37–24

37.6 Existence and Uniqueness for Single N^{th}-order Differential Equations 37–25

Exercises 37–27

Some Answers to Some of the Exercises 37–32

38 General Solutions to Homogeneous Linear Systems **38–1**

38.1 Basic Assumptions and Terminology 38–1

38.2 Deriving the Main Results 38–2

Immediate Results on Existence and Uniqueness 38–3

Linear Combinations and the Principle of Superposition 38–3

Linear Independence 38–6

Fundamental Sets of Solutions 38–8

38.3 The Main Result on General Solutions to Linear Systems 38–10

38.4 Wronskians and Identifying Fundamental Sets 38–10

A "Matrix/Vector" Formula for Linear Combinations 38–11

Deriving a 'Simple' Test 38–12

Wronskians and Identifying Fundamental Sets 38–13

38.5 Fundamental Matrices 38–15

Exercises 38–15

Some Answers to Some of the Exercises 38–20

39 Homogeneous Constant Matrix Systems, Part I **39–1**

39.1 Basics, and Some Fundamental Solutions 39–1

39.2 A Short Review of Eigenvalues and Eigenvectors (with Applications) 39–3

Eigen-Things 39–3

Finding (and Using) Eigen-Things 39–5

Notes on Finding and Using Eigenpairs 39–11

39.3 Eigenpairs and Corresponding Solutions 39–13

Sets of Solutions from Sets of Eigenpairs . . . 39–13
Issues . . . 39–14
39.4 Two-Dimensional Phase Portraits: Preliminaries . . . 39–15
The Critical Point at the Origin . . . 39–15
Symmetries . . . 39–15
39.5 Two-Dimensional Phase Portraits from Real, Complete Eigen-Sets . . . 39–16
Trajectories Corresponding to One Eigenpair . . . 39–17
Trajectories When $r_1 = r_2 \neq 0$ . . . 39–19
Trajectories When $0 < r_1 < r_2$ . . . 39–20
Trajectories When $r_1 < r_2 < 0$ . . . 39–22
Trajectories When $r_1 < 0 < r_2$ . . . 39–23
Exercises . . . 39–25
Some Answers to Some of the Exercises . . . 39–32

40 Homogeneous Constant Matrix Systems, Part II **40–1**
40.1 Solutions Corresponding to Complex Eigenvalues . . . 40–1
Eigenpairs of Real Matrices . . . 40–1
The Corresponding Real-Valued Solutions . . . 40–2
40.2 Two-Dimensional Phase Portraits with Complex Eigenvalues . . . 40–6
The General Solutions . . . 40–6
Trajectories when $\lambda = 0$ . . . 40–7
Trajectories when $\lambda > 0$ . . . 40–9
Trajectories when $\lambda < 0$ . . . 40–11
40.3 'Second Solutions' When the Set of Eigenvectors is Incomplete . . . 40–11
40.4 Two-Dimensional Phase Portraits with Incomplete Sets of Eigenvectors . . . 40–15
Trajectories when $r > 0$ . . . 40–16
Trajectories when $r < 0$ . . . 40–17
Trajectories when $r = 0$ . . . 40–17
40.5 Complete Sets of Solutions Corresponding to Incomplete Sets of Eigenvectors . 40–17
When the Geometric Multiplicity Is One . . . 40–18
When the Geometric Multiplicity Is Greater Than One . . . 40–21
40.6 Appendix for Sections 40.1 and 40.2 . . . 40–25
Linear Independence of Two Vectors . . . 40–25
Ellipticity of Trajectories . . . 40–26
Exercises . . . 40–29
Some Answers to Some of the Exercises . . . 40–34

41 Miscellaneous Topics Involving Homogeneous Constant Matrix Systems **41–1**
41.1 Phase Portraits for Large Constant Matrix Systems . . . 41–1
41.2 Shifted Constant Matrix Systems . . . 41–3
41.3 Classifying Critical Points for 2×2 Systems . . . 41–4
Stability in Constant Matrix Systems . . . 41–5
Nodes, Saddle Points, Centers and Spiral Points in General . . . 41–5
41.4 Phase Portraits for Imprecisely Known Systems . . . 41–6
$\epsilon < r_1 < r_2$ with $r_1 \not\approx r_2$ . . . 41–6
$\epsilon < r_1 < r_2$ with $r_1 \approx r_2$ . . . 41–6
$r = \lambda \pm i\omega$ with $\epsilon < \lambda$ and $\epsilon < \omega$ . . . 41–7
$r = \lambda \pm i\omega$ with $\lambda \approx 0$ and $\epsilon < \omega$ . . . 41–7

Other Cases . . . 41–7
41.5 Using Fundamental and Exponential Matrices . . . 41–8
Fundamental Matrices . . . 41–8
The Exponential Matrix . . . 41–11
Inverses and Limitations of Matrix Exponentials . . . 41–14
41.6 Using Similarity Transforms . . . 41–15
41.7 Euler Systems . . . 41–20
What Is An Euler System? . . . 41–20
Direction Fields and Trajectories . . . 41–20
Solving Euler Systems . . . 41–22
Exercises . . . 41–24
Some Answers to Some of the Exercises . . . 41–28
Private Notes for Chapter 41 . . . 41–29

42 Nonhomogeneous Linear Systems **42–1**
42.1 General Theory . . . 42–1
42.2 Method of Undetermined Coefficients / Educated Guess . . . 42–3
First Guesses . . . 42–3
Second and Subsequent Guesses . . . 42–5
42.3 Reduction of Order/Variation of Parameters . . . 42–9
The Basic Variation of Parameters Formula . . . 42–9
The Definite Integral Version . . . 42–11
42.4 Laplace Transforms . . . 42–14
42.5 Using Similarity Transforms . . . 42–15
Exercises . . . 42–18
Private Notes for Chapter 42 . . . 42–19

43 Nonlinear Autonomous Systems of Differential Equations **43–1**
43.1 The Systems of Interest and a Little Review . . . 43–1
43.2 Rewriting Systems Using Jacobian Matrices . . . 43–3
The Jacobian Matrix of a System . . . 43–3
Recollections of Differentiability . . . 43–4
Differential Form for a Vector-Valued Function . . . 43–6
43.3 Linearized Systems and Trajectories Near Critical Points . . . 43–7
43.4 Analyzing Trajectories for Nonlinear Systems . . . 43–12
43.5 Application: Competing Criters (Species) . . . 43–14
A Single Species Competing with Itself . . . 43–14
Two Competing Species . . . 43–15
General Analysis of the Competing Species Model . . . 43–21
43.6 Application: The Damped Pendulum . . . 43–26
Exercises . . . 43–30
Some Answers to Some of the Exercises . . . 43–34
Private Notes for Chapter 43 . . . 43–35

44 Level Curves and Equations for Trajectories **44–1**

45 The Liapunov Theory **45–1**

VII Boundary-Value Problems

46 Boundary-Value Problems **46–1**
46.1 Basic Second-Order Boundary-Value Problems . . . 46–1
46.2 Classes of Boundary Conditions . . . 46–3
46.3 Homogeneous and Nonhomogeneous Boundary-Value Problems . . . 46–5
Exercises . . . 46–7
Some Answers to Some of the Exercises . . . 46–10

47 Sturm-Liouville Problems **47–1**
47.1 Linear Algebraic Antecedents . . . 47–2
Recollections From (Real) Linear Algebra . . . 47–2
Linear Algebra with Complex Components . . . 47–3
Comments . . . 47–8
47.2 Boundary-Value Problems with Parameters . . . 47–9
47.3 The Sturm-Liouville Form for a Differential Equation . . . 47–12
47.4 Boundary Conditions for Sturm-Liouville Problems . . . 47–15
Green's Formula, and "Sturm-Liouville Appropriate" Boundary Conditions . . . 47–15
47.5 Sturm-Liouville Problems . . . 47–17
Full Definition . . . 47–17
A Important Class of Sturm-Liouville Problems . . . 47–18
So What? . . . 47–18
47.6 The Eigen-Spaces . . . 47–19
47.7 Inner Products, Orthogonality and Generalized Fourier Series . . . 47–20
Inner Products for Functions . . . 47–20
Norms . . . 47–21
Orthogonality . . . 47–22
Generalized Fourier Series . . . 47–23
Approximations and Completeness . . . 47–25
47.8 Sturm-Liouville Problems and Eigenfunction Expansions . . . 47–26
Self-Adjointness and Some Immediate Results . . . 47–26
Other Results Concerning Eigenvalues . . . 47–28
The Eigenfunctions . . . 47–29
47.9 The Main Results Summarized (Sort of) . . . 47–30
A Mega-Theorem . . . 47–30
Exercises . . . 47–32
Some Answers to Some of the Exercises . . . 47–40

48 Applications to PDE Problems* **48–1**
48.1 The Heat Flow Problem . . . 48–1
Setting Up the Problem . . . 48–1
A Formal Solution . . . 48–2
Validity and Properties of the Formal Solution . . . 48–5
48.2 The Vibrating String Problem . . . 48–7
Setting Up the Problem . . . 48–7
A Formal Solution . . . 48–8
Harmonics of a Vibrating String . . . 48–11
Exercises . . . 48–12

Some Answers to Some of the Exercises . . . 48–16

49 Choosing Sturm-Liouville Problems **49–1**

49.1 Separation of Variables, Slightly Streamlined . . . 49–1

The PDE Problem . . . 49–1

Separable (Partial) Solutions . . . 49–2

The Separation of Variables Procedure . . . 49–3

Continue Solving . . . 49–5

Exercises . . . 49–8

Some Answers to Some of the Exercises . . . 49–10