49

Choosing Sturm-Liouville Problems

We just saw examples of how the Sturm-Liouville theory can be used to create infinite series solutions to some problems involving partial differential equations. What was not explained was how you chose the relevant Strum-Liouville problem.

49.1 Separation of Variables, Slightly Streamlined

What we are about to describe is the basic "separation of variables" method for solving certain partial differential equation problems. It starts by breaking down the given partial differential equation problem to a pair of ordinary differential differential equation problems, one of which turns out to be a Sturm-Liouville problem. This method is a cornerstone in the study of partial differential equations, and is a major element in many introductory courses on partial differential equations.

The PDE Problem

We will assume that we have some problem in which there are two basic variables. For convenience, we will denote these variables as t and x. Our interest is in finding the function $u = u(x, t)$ that satisfies some given partial differential equation along with some other "boundary and initial conditions". If x and t denote "position" and "time", as is traditional, then we are typically seeking the solution $u = u(x, t)$ when $a < x < b$ for some interval (a, b) and $t > 0$. The "boundary conditions for $u(x, t)$" are specifications on either u or $\partial u/\partial x$ at the endpoints of the interval, $x = a$ and $x = b$, and the "initial conditions" will be specification on either u or $\partial u/\partial t$ when $t = 0$.

!►*Example 49.1:*

`The Heat Flow Problem in the previous chapter`

!►*Example 49.2:*

$$\frac{\partial u}{\partial t} - x^2 \frac{\partial^2 u}{\partial x^2} - x \frac{\partial u}{\partial x} = 0 \qquad \text{for} \quad 1 < x < e^{\pi} \tag{49.1}$$

with boundary conditions

$$u(1,t) = 0 \qquad , \qquad u(e^{\pi},t) = 0 \tag{49.2}$$

and initial conditions

$$u(x,0) = f(x) \qquad \text{for} \quad 1 < x < e^{\pi} \tag{49.3}$$

for some known function f *.*

For this approach to work, the boundary conditions must correspond to "Sturm-Liouville appropriate" boundary conditions. There are also certain restrictions on the partial differential equation. One of those is that the partial differential equation is a second-order homogeneous partial differential equation. (Though it can be extended to deal with higher-order and nonhomogeneous partial differential equations.)

Separable (Partial) Solutions

Let us suppose we have a suitable pde problem as just described, say, the one described in example 49.2, and we want to solve it using an eigenfunction expansion corresponding to an appropriate Sturm-Liouville problem. That means we can write our solution as

$$u(x,t) = \sum_k b_k(t)\phi_k(x) \tag{49.4}$$

where the $\phi_k(x)$'s are the yet unknown eigenfunctions from our yet unknown Sturm-Liouville problem, and

$$b_k(t) = \frac{\langle\, \phi_k(x) \mid u(x,t) \,\rangle}{\|\phi_k\|^2}$$

using, of course, the inner product associated with our yet unknown Sturm-Liouville problem. For convenience, let's rewrite our series formula for $u(x,t)$ as

$$u(x,t) = \sum_k u_k(x,t) \qquad \text{with} \quad u_k(x,t) = b_k(t)\phi_k(x) \quad .$$

In our examples in the previous chapter, each $u_k(x,t) = b_k(t)\phi_k(x)$ satisfied both the boundary condition and the partial differential equation. That is what we want in general. In a way, this briefly reduces our problem to that of finding every

$$u_k(x,t) = b_k(t)\phi_k(x)$$

satisfying both the partial differential equation and the boundary conditions of our big problem. Because these u_k's will not, themselves, satisfy the initial conditions, we'll just call these *partial solutions* to our pde problem. To find formulas for these partial solutions, we'll use a procedure that "separates" our problem into two related problems — one involving $b_k(t)$ and a Sturm-Liouville problem involving $\phi_k(t)$. Because of this , we further say these are *separable* partial solutions.

The method for separating our problem into these two related problems is called "separation of variables".

The Separation of Variables Procedure

We now consider the problem of finding every

$$u_k(x,t) = b(t)\phi(x)$$

satisfying our given partial differential equation and boundary conditions. We're dropping the indices on the $b(t)$ and $\phi(x)$ because, frankly, they will just get in the way for now. Later, we'll resurrect the appropriate indices.

Here, now, are the steps in doing "separation of variables":

1. Assume

$$u_k(x,t) = \phi(x)b(t) \quad .$$

2. Identify the boundary conditions, plug $u_k(x,t) = \phi(x)b(t)$ into those conditions, and determine the corresponding boundary conditions for ϕ.

In our example, the boundary conditions are

$$u(1,t) = 0 \quad \text{and} \quad u(e^{\pi},t) = 0 \quad \text{for} \quad 0 < t \quad .$$

Replacing $u(x,t)$ *with* $\phi(x)b(t)$ *in the first gives*

$$\phi(1)b(t) = 0 \quad \text{for} \quad 0 < t \quad .$$

This means that either

$$\phi(1) = 0 \quad \text{or} \quad b(t) \equiv 0 \quad .$$

If $b(t) \equiv 0$*, then we get*

$$u(x,t) = \phi(x)b(t) = \phi(x) \cdot 0 = 0 \quad ,$$

which not very interesting. So, to avoid just getting the trivial solution to our partial differential equation, we will instead require that

$$\phi(1) = 0 \quad .$$

Plugging $u(x,t) = \phi(x)b(t)$ *in the second boundary condition gives*

$$\phi(e^{\pi})b(t) = 0 \quad \text{for} \quad 0 < t \quad .$$

Hence, again,

$$\phi(e^{\pi}) = 0 \quad .$$

Thus, our $\phi(x)$ *must, itself, satisfy the two boundary conditions*

$$\phi(1) = 0 \quad \text{and} \quad \phi(e^{\pi}) = 0 \quad . \tag{49.5}$$

3. Plug $u_k(x,t) = \phi(x)b(t)$ into the partial differential equation, simplify and rearrange (using good algebra) to get

$$\text{formula of } t \text{ only} = \text{formula of } x \text{ only} \quad . \tag{49.6}$$

Some side notes:

(a) Dividing through by $\phi(x)b(t)$ is usually a good idea.

(b) For reasons that won't be clear for a while, it is usually a good idea to move any 'floating' constants to the side with the t variable.

(c) If you can get the equation into form (49.6), then your partial differential equation is said to be *separable*. Otherwise, the partial differential equation is "not separable" and this approach leads to a disappointing dead end. For the rest of this discussion, we will assume the equation *is* separable.

In our example, with $u_k(x,t) = \phi(x)b(t)$,

$$\frac{\partial u}{\partial t} - x^2 \frac{\partial^2 u}{\partial x^2} - x \frac{\partial u}{\partial x} = 0$$

$$\hookrightarrow \quad \frac{\partial}{\partial t}[\phi(x)b(t)] - x^2 \frac{\partial^2}{\partial x^2}[\phi(x)b(t)] - x \frac{\partial}{\partial x}[\phi(x)b(t)] = 0$$

$$\hookrightarrow \quad \phi(x)b'(t) - x^2\phi''(x)b(t) - x\phi'(x)b(t) = 0$$

$$\hookrightarrow \quad \frac{\phi(x)b'(t) - x^2\phi''(x)b(t) - x\phi'(x)b(t)}{\phi(x)b(t)} = \frac{0}{\phi(x)b(t)}$$

$$\hookrightarrow \quad \frac{b'(t)}{b(t)} - \frac{x^2\phi''(x) + x\phi'(x)}{\phi(x)} = 0 \quad .$$

Hence,

$$\frac{b'(t)}{b(t)} = \frac{x^2\phi''(x) + x\phi'(x)}{\phi(x)} \quad .$$

4. "Observe" that the only way we can have

$$\text{formula of } t \text{ only} = \text{formula of } x \text{ only}$$

is for both sides to be equal to a single constant. Because it usually simplifies later work slightly, we will denote this constant by " $-\lambda$ ". This λ is the *separation constant* and is totally unknown at this point (and, yes, the values for λ will also be the eigenvalues for the Sturm-Liouville problem we'll derive).

After making this observation:

(a) Write this fact down using the separation constant.

(b) Observe further that this gives us a pair of *ordinary* differential equations, each involving λ .

(c) Write out that pair of ordinary differential equations in as simplified a form as seems reasonable.

Continuing our example, we have

$$\frac{b'(t)}{b(t)} = \frac{x^2\phi''(x) + x\phi'(x)}{\phi(x)} = -\lambda \quad .$$

That is,

$$\frac{b'(t)}{b(t)} = -\lambda \qquad \text{and} \qquad \frac{x^2\phi''(x) + x\phi'(x)}{\phi(x)} = -\lambda \quad .$$

After a little algebra, these become

$$b'(t) = -\lambda b(t) \qquad \text{and} \qquad x^2\phi''(x) + x\phi'(x) = -\lambda\phi(x) \quad .$$

(Keep in mind that the two ordinary differential equations are not independent of each other — they are linked by the common value λ .)

5. Under the ordinary differential equation for ϕ , write out the boundary conditions obtained for ϕ in step 2.

   For our example, the boundary conditions for ϕ are given by equation set (49.5). Placing them with the ordinary differential equation for ϕ gives

$$b'(t) = -\lambda b(t) \qquad\qquad \begin{aligned} x^2\phi''(x) + x\phi'(x) &= -\lambda\phi(x) \\ \phi(1) &= 0 \\ \phi(e^\pi) &= 0 \end{aligned}$$

Technically, we're now finished with "separation of variables". Using it, we have decomposed our "partial differential equation with boundary conditions" into two ordinary differential equation problems. These two problems are related to each other only through the common constant λ . Observe that the ordinary differential equation problem with the boundary conditions is a *Sturm-Liouville problem*, and that the possible eigenvalues are also the possible values for the separation constant (if the sign on the λ is 'wrong', just go back to step 4 and change the sign in front of λ).

The other ordinary differential equation will be officially called *the other problem*.

Continue Solving

Obviously, the next steps are to finish solving for the $\phi_k(x)$'s and corresponding $b_k(t)$'s , and to use them to finish solving our original partial differential equation problem. We'll continue our steps where we left off.

6. "Solve the Sturm-Liouville problem." More precisely, take the Sturm-Liouville problem and find

   (a) the corresponding weight function $w(x)$ and inner product

$$\langle\, f \mid g \,\rangle = \int_a^b \overline{f(x)}\, g(x) w(x)\, dx \quad ,$$

   (b) a corresponding complete orthogonal set of eigenfunctions

$$\{\, \phi_0(x) \,,\, \phi_1(x) \,,\, \phi_2(x) \,,\, \dots \,\} \quad ,$$

   and

   (c) the corresponding set of eigenvalues

$$\{\, \lambda_0 \,,\, \lambda_1 \,,\, \lambda_2 \,,\, \dots \,\} \quad .$$

   This, of course, assumes that the Sturm-Liouville problem is either a regular Sturm-Liouville problem, so that theorem 47.11 on page 47–30 holds, or is one of the other Sturm-Liouville problems for which a similar theorem holds.

In our example, the Sturm-Liouville problem is

$$x^2\phi''(x) + x\phi'(x) = -\lambda\phi(x)$$

with

$$\phi(1) = 0 \quad \text{and} \quad \phi(e^\pi) = 0 \quad .$$

The equation just happens the same one in exercise 47.9 c on page 47–33. The Sturm-Liouville form for this equation is

$$\frac{d}{dx}\left[x\frac{d\phi}{dx}\right] = -\lambda\frac{1}{x}\phi \quad ,$$

(as you derived in exercise 47.9 c). The associated weight function is

$$w(x) = \frac{1}{x} = x^{-1}$$

and the corresponding inner product is given by

$$\langle f \mid g \rangle = \int_1^{e^\pi} \overline{f(x)}g(x)x^{-1}\,dx \quad .$$

This is a regular Sturm-Liouville problem. So we know there is a complete orthogonal set of eigenfunctions

$$\{\phi_1(x), \phi_2(x), \phi_3(x), \ldots\}$$

with the corresponding set of eigenvalues

$$\{\lambda_1, \lambda_2, \lambda_3, \ldots\} \quad .$$

In fact, if you look at the answers to exercise 47.11 b on page 47–34, you will see that all the eigenvalues are given by

$$\lambda_k = k^2 \quad \text{for} \quad k = 1, 2, 3, \ldots$$

and for the corresponding ϕ_k*'s, we can use*

$$\phi_k(x) = \sin(k \ln|x|) \quad \text{for} \quad k = 1, 2, 3, \ldots$$

7. Now solve the other problem for each λ_k, obtaining the corresponding $b_k(t)$'s.

   In our example, the other problem is

$$b'(t) = -\lambda b(t) \quad .$$

   No matter what real value λ *is, the general solution to this differential equation is*

$$b(t) = Be^{-\lambda t}$$

   where B *is an arbitrary constant. So, for each* λ_k *found above,*

$$b_k(t) = B_k e^{-\lambda_k t} = B_k e^{-k^2 t}$$

   where B_k *is an arbitrary constant.*

8. For each eigenvalue λ_k, write out the corresponding separable solution to the partial differential equation,

$$u_k(x,t) = \phi_k(x)b_k(t) \quad ,$$

combining arbitrary constants as appropriate. (It's also a good idea to state the range for the indexing.)

For our example,

$$u_k(x,t) = \phi_k(x)b_k(t) = B_k e^{-k^2 t}\sin(k\ln|x|) \qquad \text{for} \quad k = 1, 2, 3, \ldots \quad .$$

9. Write out

$$u(x,t) = \sum_k \phi_k(x)b_k(t)$$

Since each $u_k(x,t) = \phi_k(x)b_k(t)$ satisfies the partial differential equation and boundary conditions, so will the entire series. All that remains is to determine the remaining constant in each $\phi_k(x)b_k(t)$ so that the series also satisfies the given initial conditions. Use the fact that the ϕ_k's form a complete orthogonal set with respect to the weight function found above.

For our example,

$$u(x,t) = \sum_{k=1}^{\infty} B_k e^{-k^2 t}\sin(k\ln|x|) \quad .$$

Using this with the give initial condition:

$$f(x) = u(x,0) = \sum_{k=1}^{\infty} B_k e^{-k^2 \cdot 0}\sin(k\ln|x|) \quad ,$$

which simplifies to

$$f(x) = \sum_{k=1}^{\infty} B_k \sin(k\ln|x|)$$

Since the right side must be the eigenfunction expansion of f, we must have

$$B_k = \frac{\langle \phi_k \mid f \rangle}{\|\phi_k\|} = \frac{\int_1^{e^\pi} \sin(k\ln|x|)\, f(x)x^{-1}\,dx}{\int_1^{e^\pi} |\sin(k\ln|x|)|^2\, x^{-1}\,dx} \quad .$$

With a little work, you can show the integral on the bottom equals $\pi/2$, and, hence,

$$B_k = \frac{2}{\pi}\int_1^{e^\pi} f(x)\sin(k\ln|x|)\,x^{-1}\,dx \quad .$$

10. Finally, summarize the results derived and write out the final series formula for $u(x,t)$.

For our example,

$$u(x,t) = \sum_{k=1}^{\infty} B_k e^{-k^2 t}\sin(k\ln|x|)$$

where

$$B_k = \frac{2}{\pi}\int_1^{e^\pi} f(x)\sin(k\ln|x|)\,x^{-1}\,dx \quad .$$

Additional Exercises

49.1. *Consider the partial differential equation problem*

$$\frac{\partial^2 u}{\partial t^2} - 9\frac{\partial^2 u}{\partial x^2} = 0 \quad \text{for} \quad 0 < x < 1$$

with boundary conditions

$$u(0,t) = 0 \quad \text{and} \quad u(1,t) = 0 \quad ,$$

and initial conditions

$$u(x,0) = f(x) \quad \text{and} \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0 \quad .$$

Using the separation of variables method as appropriate:

a. *Find the appropriate Sturm-Liouville problem.*

b. *Find the corresponding "other problem" (for the $b_k(t)$'s).*

c. *What is the appropriate inner product $\langle f \mid g \rangle$ for the eigenfunctions in the associated Sturm-Liouville problem.*

49.2. *Consider the partial differential equation problem*

$$\frac{\partial u}{\partial t} - x^2\frac{\partial^2 u}{\partial x^2} - 3\frac{\partial u}{\partial x} = 0 \quad \text{for} \quad 1 < x < e^{\pi}$$

with boundary conditions

$$u(1,t) = 0 \quad \text{and} \quad u(e^{\pi},t) = 0 \quad ,$$

and initial condition

$$u(x,0) = f(x) \quad .$$

Using the separation of variables method as appropriate:

a. *Find the appropriate Sturm-Liouville problem.*

b. *Find the corresponding "other problem" (for the $b_k(t)$'s).*

c. *What is the appropriate inner product $\langle f \mid g \rangle$ for the eigenfunctions in the associated Sturm-Liouville problem.*

49.3. *Consider the partial differential equation problem*

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} - 2\frac{\partial u}{\partial x} = 0 \quad \text{for} \quad 0 < x < \text{‘}$$

with boundary conditions

$$u(0,t) = 0 \quad \text{and} \quad \left.\frac{\partial u}{\partial x}\right|_{x=1} = 0 \quad ,$$

and initial condition

$$u(x,0) = f(x) \quad .$$

Using the separation of variables method as appropriate:

a. *Find the appropriate Sturm-Liouville problem.*

b. *Find the corresponding "other problem" (for the* $b_k(t)$ *'s).*

c. *What is the appropriate inner product* $\langle f \mid g \rangle$ *for the eigenfunctions in the associated Sturm-Liouville problem.*

49.4. *Finish finding the series solution to the partial differential equation problem in exercise 49.2 for an arbitrary* $f(x)$ *.*

Some Answers to Some of the Exercises

WARNING! Most of the following answers were prepared hastily and late at night. They have not been properly proofread! Errors are likely!

1a. $\phi''(x) = -\lambda\phi(x)$ for $0 < x < 1$ with $\phi(0) = 0$ and $\phi(1) = 0$

1b. $b''(t) + 9\lambda b(t) = 0$

1c. $\displaystyle\int_1^{e^\pi} \overline{f(x)} g(x)\, dx$

2a. $x^2\phi''(x) + 3x\phi'(x) = -\lambda\phi(x)$ for $1 < x < e^\pi$ with $\phi(1) = 0$ and $\phi(e^\pi) = 0$

2b. $b''(t) + \lambda b(t) = 0$

2c. $\displaystyle\int_1^{e^\pi} \overline{f(x)} g(x) x\, dx$

3a. $\phi''(x) + 2\phi'(x) = -\lambda\phi(x)$ for $0 < x < 1$ with $\phi(1) = 0$ and $\phi'(1) = 0$

3b. $b''(t) + \lambda b(t) = 0$

3c. $\displaystyle\int_1^{e^\pi} \overline{f(x)} g(x) e^{2x}\, dx$

4. $\displaystyle u(x,t) = \sum_{k=1}^{\infty} B_k e^{-(k^2+1)t} \frac{\sin(k \ln|x|)}{x}$ with $B_k = \frac{\pi}{2} \int_1^{e^\pi} f(x) \sin(k \ln|x|)\, dx$