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Boundary-Value Problems

We are now switching back to problems involving solitary differential equations. For the most part, these differential equations will be linear and second order. And, for various reasons, we will switch back to using x instead of t as the variable. One thing that will distinguish these problems from others we've explored is that, instead of having "initial conditions" placed on the solution and some of its derivatives at a single point, we will have constraints placed on the solution and/or some of its derivatives at two distinct points. These constraints are called "boundary values" because, typically, the two points make up the boundary of the interval of interest.

Truth is, we will be able to say just about all there is to say about mere boundary-value problems in a rather short space. And we won't do much to motivate our study of boundary-value problems. They do arise in applications, but these applications are not quickly developed. In fact, the main applications are boundary-value problems that arise in the study of partial differential equations, and those boundary-value problems also involve "eigenvalues". We will start studying this rather important class of boundary-value problems in the next chapter using material developed in this chapter.

46.1 Basic Second-Order Boundary-Value Problems

A *second-order boundary-value problem* consists of a second-order differential equation along with constraints on the solution $y = y(x)$ at two values of x. For example,

$$y'' + y = 0 \quad \text{with} \quad y(0) = 0 \quad \text{and} \quad y(\pi/6) = 4$$

is a fairly simple boundary value problem. So is

$$y'' + y = 0 \quad \text{with} \quad y'(0) = 0 \quad \text{and} \quad y'(\pi/6) = 4 \quad .$$

Alternatively, we might not actually require particular values at the two points, just that they are related in some way. For example:

$$y'' + y = 0 \quad \text{with} \quad y(0) = y(\pi/6) \quad \text{and} \quad y'(0) = y'(\pi/6) \quad .$$

The constraints given at the two points are called either *boundary values* or *boundary conditions*. Typically, the interval of interest for the differential equation is the interval between the two points at which boundary conditions are specified. Hence, these two points are often referred to as *boundary points*.

None-too-surprisingly, a *solution* to a given boundary-value problem is a function that satisfies the given differential equation over the interval of interest, along with as the given boundary conditions. We will discuss just what are "appropriate" boundary conditions in the next section. For now, let us solve a few boundary-value problems involving the differential equation

$$y'' + y = 0 \quad .$$

Recall that the general solution to this equation is

$$y(x) = c_1 \cos(x) + c_2 \sin(x) \quad .$$

So the only work in solving these boundary-value problems is in determining the values of c_1 and c_2 so that the above formula (with the determined values of c_1 and c_2) satisfies the boundary conditions.

!►Example 46.1: *We start with*

$$y'' + y = 0 \qquad \text{with} \quad y(0) = 0 \quad \text{and} \quad y\left({}^{\pi}\!/_{6}\right) = 4 \quad .$$

Combining the general solution of the differential equation with the boundary conditions yields the system

$$\begin{aligned} 0 &= y(0) = c_1 \cos(0) + c_2 \sin(0) = c_1 \cdot 1 + c_2 \cdot 0 \\ 4 &= y(1) = c_1 \cos\left(\frac{\pi}{6}\right) + c_2 \sin\left(\frac{\pi}{6}\right) = c_1 \cdot \frac{\sqrt{3}}{2} + c_2 \cdot \frac{1}{2} \end{aligned} \quad .$$

From the first equation we get

$$c_1 = 0 \quad .$$

Thus, the second equation reduces to

$$4 = 0 \cdot \frac{\sqrt{3}}{2} + c_2 \cdot \frac{1}{2} = \frac{1}{2} c_2 \quad .$$

Hence,

$$c_1 = 0 \qquad \text{and} \qquad c_2 = 8 \quad ,$$

and the one and only solution to our boundary-value problem is

$$y(x) = 8 \sin(x) \quad .$$

!►Example 46.2: *Now consider*

$$y'' + y = 0 \qquad \text{with} \quad y(0) = 0 \quad \text{and} \quad y(\pi) = 4 \quad .$$

Again, we combine the general solution of the differential equation with the boundary conditions, this time obtaining the system

$$\begin{aligned} 0 &= y(0) = c_1 \cos(0) + c_2 \sin(0) = c_1 \cdot 1 + c_2 \cdot 0 = c_1 \\ 4 &= y(\pi) = c_1 \cos(\pi) + c_2 \sin(\pi) = c_1 \cdot (-1) + c_2 \cdot 0 = -c_1 \end{aligned} \quad .$$

So, the single constant c_1 *must satisfy both*

$$c_1 = 0 \quad \text{and} \quad c_1 = -4 \quad .$$

But this is impossible. Hence, a solution to the given boundary-value problem is not possible. There is no solution.

!▶Example 46.3: *Finally, for now, consider*

$$y'' + y = 0 \quad \text{with} \quad y(0) = 0 \quad \text{and} \quad y(\pi) = 0 \quad .$$

Once again, we combine the general solution of the differential equation with the boundary conditions, this time obtaining

$$0 = y(0) = c_1 \cos(0) + c_2 \sin(0) = c_1 \cdot 1 + c_2 \cdot 0 = c_1$$
$$0 = y(\pi) = c_1 \cos(\pi) + c_2 \sin(\pi) = c_1 \cdot (-1) + c_2 \cdot 0 = -c_1 \quad .$$

Both of these equations reduce to

$$c_1 = 0 \quad ,$$

which specifies c_1, *but says nothing about* c_2, *leaving us with*

$$y(x) = c_2 \sin(x)$$

where c_2 *can be any constant. And, indeed, you can easily verify that this satisfies the differential equation and the given boundary conditions for our problem.*

Consequently, this boundary-value problem does not merely have a solution — it has infinitely many solutions, one for each different value of c_2.

Glance back over the three examples above. The given boundary-value problems differed only in relatively small details, and yet we ended up with three radically different results: a single solution in the first example, no solution in the second, and infinitely many solutions in the last. This is a feature of boundary-value problems — any given boundary-value problem may have either one solution, no solutions or many solutions.

46.2 Classes of Boundary Conditions

While many different types of "boundary conditions" can be invented, there are only three that arise in practice often enough to be of interest to us. They are the "regular", "boundedness" and "periodic" boundary conditions:

Regular boundary conditions: A boundary condition at $x = x_0$ is said to be *regular* if and only if it can be described by

$$\alpha y(x_0) + \beta y'(x_0) = \gamma$$

where α, β and γ are constants, with α or β (or both) being nonzero.

In practice, either β or α is often zero, in which case the above reduces to

$$y(x_0) = \gamma \quad \text{or} \quad y'(x_0) = \gamma \quad .$$

And in many cases, $\gamma = 0$.

Boundedness boundary conditions: This is where we simply say that a solution does not "blow up" at a point $x = x_0$. To be precise,

$$\lim_{x \to x_0} |y(x)| < \infty \quad .$$

In practice, we usually write this as

$$|y(x_0)| < \infty \quad .$$

Such conditions are typically the appropriate conditions when x_0 is a singular point for the differential equation. (Forgot what a singular point is? Glance back at page 31–28.)

Periodic boundary conditions: A *periodic* boundary condition states that the solution or its derivatives at two distinct points $x = x_0$ and $x = x_1$ are equal; that is,

$$y(x_0) = y(x_1) \qquad \text{or} \qquad y'(x_0) = y'(x_1) \quad .$$

In practice, these two periodic boundary conditions often occur together.

These conditions naturally occur when the variable x is actually the angular component θ in a polar coordinate system.

The first two types of boundary condtions, regular and boundedness, are often said to be *separated* boundary conditions, since they can be imposed separately at each boundary point.

!►Example 46.4: *Here is a boundary-value problem with one boundedness condition (at $x = 0$ and one regular boundary condition:*

$$x^2 y'' + xy' - 4y = 0 \qquad \text{with} \quad |y(0)| < \infty \quad \text{and} \quad y'(1) = 6 \quad .$$

The differential equation is an Euler equation. Plugging $y = x^r$, we get

$$x^2 \left[r(r-1)x^{r-2} \right] + x \left[r x^{r-1} \right] - 4x^r = 0$$

$$\hookrightarrow \qquad r^2 - 4 = 0$$

$$\hookrightarrow \qquad r = \pm 2 \quad .$$

So the general solution to the differential equation is

$$y(x) = c_1 x^2 + c_2 x^{-2} \quad .$$

Using this with the boundedness condition at $x = 0$, we get

$$\begin{aligned} \infty &> \lim_{x \to 0} |y(x)| \\ &= \lim_{x \to 0} \left| c_1 x^2 + c_2 x^{-2} \right| \\ &= \lim_{x \to 0} \left| 0 + c_2 x^{-2} \right| = \begin{cases} +\infty & \text{if} \quad c_2 \neq 0 \\ 0 & \text{if} \quad c_2 = 0 \end{cases} \quad . \end{aligned}$$

Hence, the boundedness condition forces c_2 *to be zero. That leaves us with*

$$y(x) = c_1 x^2 \quad .$$

which then means that $y'(x) = c_1 2x$ *. So, to satisfy the boundary condition at* $x = 1$ *, we must have*

$$6 = y'(1) = c_1 2 \cdot 1 = 2c_1 \quad .$$

Thus, $c_1 = 3$ *, and the single solution to our boundary-value problem is*

$$y(x) = 3x^2 \quad .$$

46.3 Homogeneous and Nonhomogeneous Boundary-Value Problems

Recall that a linear, second-order differential equation

$$ay'' + by' + cy = g$$

is said to be homogeneous if and only if $g = 0$. Way back when we first studied them, we saw that the solutions to these equations satisfied the "principle of superposition". That is, if

$$y_1(x) \qquad \text{and} \qquad y_2(x)$$

are both solutions to the same homogeneous differential equation, then so is any linear combination

$$y(x) = c_1 y_1(x) + c_2 y_2(x) \quad .$$

In much of our future work, it will be important that a similar "principle of superposition" holds for solutions to our boundary-value problems. For convenience, let us refer to a given boundary condition as being *homogeneous* if and only if the following condition holds:

> *Whenever* $y_1(x)$ *and* $y_2(x)$ *satisfies that boundary condition, so does any linear combination*
>
> $$y(x) = c_1 y_1(x) + c_2 y_2(x) \quad .$$

Let us go back through our types of boundary conditions, and see which are homogeneous:

Regular boundary conditions: Suppose both $y_1(x)$ and $y_2(x)$ satisfy

$$\alpha y_k(x_0) + \beta {y_k}'(x_0) = \gamma$$

for some constants α , β and γ , with α or β (or both) being nonzero. Does

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

also satisfy

$$\alpha y(x_0) + \beta y'(x_0) = \gamma \qquad (46.1)$$

for every choice of constants c_1 and c_2 ? Well,

$$\begin{aligned}\alpha y(x_0) + \beta y'(x_0) &= \alpha\left[c_1 y_1(x_0) + c_2 y_2(x_0)\right] + \beta\left[c_1 y_1{}'(x_0) + c_2 y_2{}'(x_0)\right] \\ &= c_1\left[\alpha y_1(x_0) + \beta y_1{}'(x_0)\right] + c_2\left[\alpha y_2(x_0) + \beta y_2{}'(x_0)\right] \\ &= c_1\gamma + c_2\gamma \\ &= (c_1 + c_2)\gamma \quad .\end{aligned}$$

Hence, we have equation (46.1) holding if and only if

$$(c_1 + c_2)\gamma = \gamma$$

for every pair of constants c_1 and c_2 , which is possible only if $\gamma = 0$.

In summary, a boundary condition at x_0 is a *homogeneous regular boundary condition* if and only if it can be described by

$$\alpha y(x_0) + \beta y'(x_0) = 0$$

where α and β are constants, with at least one being nonzero.

Boundedness boundary conditions: Suppose both $y_1(x)$ and $y_2(x)$ satisfy

$$|y_k(x_0)| < \infty \quad .$$

That is

$$\lim_{x\to x_0} |y_1(x)| < \infty \qquad \text{and} \qquad \lim_{x\to x_0} |y_2(x)| < \infty \quad .$$

Letting

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

we see that

$$\begin{aligned}\lim_{x\to x_0} |y(x)| &= \lim_{x\to x_0} |c_1 y_1(x) + c_2 y_2(x)| \\ &\leq |c_1| \lim_{x\to x_0} |y_1(x)| + |c_2| \lim_{x\to x_0} |y_2(x)| < \infty \quad .\end{aligned}$$

So any boundedness boundary condition is automatically homogeneous.

Periodic boundary conditions: Periodic boundary conditions are also automatically homogeneous. You can verify it yourself (see exercise 46.2).

Let us now define a *homogeneous boundary-value problem* to be a boundary-value problem consisting of a homogeneous linear differential equation along with only homogeneous boundary conditions. Combining the superposition principle for homogeneous linear differential equations and our definition of homogeneous boundary conditions gives us the superposition principle for homogeneous boundary-value problems.

Theorem 46.1 (principle of superposition for homogeneous boundary-value problems)
Any linear combination of solutions to a homogeneous boundary-value problem is, itself, a solution to that homogeneous boundary-value problem.

Let us suppose that we have a homogeneous boundary-value problem, and that $y(x)$ is a nontrivial solution (i.e., $y(x)$ is not the constant function 0 on the interval of interest). The principle of superposition tells us that $cy(x)$ is also a solution, no matter what value we use for the constant c. Since we have an infinite number of possible values for this c, we have the following corollary of the principle of superposition.

Corollary 46.2

Any homogeneous boundary-value problem has either no solutions, just the constant solution $y = 0$, or an infinite number of solutions.

Additional Exercises

46.1. *Several boundary-value problems are given below. Attempt to find all solutions to each, and state whether the problem has no solutions, one solution or infinitely many solutions.*

a. $y'' + 9y = 0$ *with* $y(0) = 0$ *and* $y\left(\frac{\pi}{2}\right) = 6$

b. $y'' + 9y = 0$ *with* $y'(0) = 0$ *and* $y'\left(\frac{\pi}{2}\right) = 6$

c. $y'' + 9y = 0$ *with* $y'(0) = 0$ *and* $y'\left(\frac{\pi}{2}\right) = 0$

d. $y'' + 9y = 0$ *with* $y(0) = 0$ *and* $y(\pi) = 6$

e. $y'' + 9y = 0$ *with* $y(0) = 0$ *and* $y'\left(\frac{\pi}{2}\right) = 0$

f. $y'' + y = 0$ *with* $y(0) = y(\pi)$

g. $y'' + y = 0$ *with* $y(0) = y(\pi)$ *and* $y'(0) = y'(\pi)$

h. $y'' + y = 0$ *with* $y(0) = y(2\pi)$ *and* $y'(0) = y'(2\pi)$

i. $y'' - 9y = 0$ *with* $y(0) = 0$ *and* $y(1) = 6$

j. $y'' - 9y = 0$ *with* $y(0) = 0$ *and* $y(1) = 0$

k. $x^2y'' - 5xy' + 8y = 0$ *with* $y(1) = 0$ *and* $y(2) = 24$

l. $x^2y'' - 2y = 0$ *with* $|y(0)| < \infty$ *and* $y'(1) = 2$

m. $x^2y'' + 3xy' = 0$ *with* $|y(0)| < \infty$ *and* $y(1) = 2$

n. $x^2y'' + 3xy' = 0$ *with* $|y(0)| < \infty$ *and* $y'(1) = 0$

46.2. *In these two problems, you show that the periodic boundary conditions are homogenous boundary conditions. In each, x_0 and x_1 are two different points on the X–axis.*

a. *Assume $y_1(x)$ and $y_2(x)$ both satisfy*

$$y_k(x_0) = y_k(x_1) \quad .$$

Show that,

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

also satisfies

$$y(x_0) = y(x_1)$$

no matter what values c_1 *and* c_2 *are.*

b. *Assume* $y_1(x)$ *and* $y_2(x)$ *both satisfy*

$$y_k{}'(x_0) = y_k{}'(x_1) \quad .$$

Show that,

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

also satisfies

$$y'(x_0) = y'(x_1)$$

no matter what values c_1 *and* c_2 *are.*

Some Answers to Some of the Exercises

WARNING! Most of the following answers were prepared hastily and late at night. They have not been properly proofread! Errors are likely!

1a. one solution: $y(x) = -6\sin(3x)$
1b. one solution: $y(x) = 2\cos(3x)$
1c. one solution: $y(x) = 0$
1d. no solution
1e. infinitely many solutions: $y(x) = c\sin(3x)$
1f. infinitely many solutions: $y(x) = c\sin(x)$
1g. no solution
1h. infinitely many solutions: $y(x) = c_1\cos(x) + c_2\sin(x)$
1i. one solution: $y(x) = 6\left[e^{3x} - e^{-3x}\right]\left[e^{3x} - e^{-3x}\right]^{-1}$
1j. one solution: $y(x) = 0$
1k. one solution: $y = 2[x^4 - x^2]$
1l. one solution: $y = x^2$
1m. one solution: $y = 2$
1n. infinitely many solutions: $y = c_1$