48

Applications to PDE Problems*

The use of the Sturm-Liouville theory can be nicely illustrated by solving any of a number of classical problems. We will consider two: a heat flow problem and a vibrating string problem. The first, determining the temperature distribution throughout some heat conducting rod, is basically the same problem Fourier first solved using "Fourier series". In some ways, this can be considered the historical starting point of the development of the Sturm-Liouville theory. The second is the problem of modeling the motion of an elastic string stretched between two points, a problem undoubtedly of interest to all guitar and banjo players.

48.1 The Heat Flow Problem

Setting Up the Problem

Here is the problem: We have a heat conducting rod of length L, and we want to know how the temperature at different points in the rod varies with time. To keep our discussion relatively simple, we assume the rod is one-dimensional, uniform, positioned along the X–axis with endpoints at $x = 0$ and $x = L$, and with the endpoints being kept at a temperature of 0 degrees (Fahrenheit, Celsius, Kelvin — the actual scale is irrelevant for us). Let

$$u(x,t) = \text{temperature of the rod's material at horizontal position } x \text{ and time } t .$$

The endpoint conditions can then be written as

$$u(0,t) = 0 \qquad \text{and} \qquad u(L,t) = 0 \qquad \text{for all} \quad t \quad .$$

Let's also assume the rod's initial temperature distribution is known; that is, we assume

$$u(x,0) = f(x)$$

where f is some known function on $(0, L)$. Let us further assume that f is at least piecewise smooth on the interval.

* Most of this chapter was stolen from chapter 16 of *Principles of Fourier Analysis* by Howell (he won't mind). I've cut out irrelevant stuff and made minor changes to fit our discussion.

If this were a text on thermodynamics or partial differential equations, we would now derive the heat equation. But this isn't such a text, so we'll simply assume that, at every point in the rod,

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2}$$

where κ is some positive constant describing the thermal properties of the rod's material. This is the famous heat equation derived by Fourier.[1]

Our goal is to find a usable formula for $u(x,t)$. Since it only makes sense to talk about the temperature where the rod exists, x must be between 0 and L. Gathering all the assumptions from above, we find that $u(x,t)$ must satisfy the following system of equations:

$$\frac{\partial u}{\partial t} - \kappa \frac{\partial^2 u}{\partial x^2} = 0 \qquad \text{for} \quad 0 < x < L \tag{48.1a}$$

$$u(0,t) = 0 \qquad \text{and} \qquad u(L,t) = 0 \tag{48.1b}$$

$$u(x,0) = f(x) \qquad \text{for} \quad 0 < x < L \tag{48.1c}$$

Implicit in this is the requirement that $u(x,t)$ be a sufficiently smooth function of x and t for the above equations to make sense. Remember, κ is a positive constant, and f is a known piecewise smooth function on $(0,L)$. At this point we have no reason to place any limits on t other than it must be real valued. So, for now, we will assume no other limits on t. Later, however, we may need to modify that assumption.[2]

A Formal Solution

Solving this problem starts with the rather bold assumption that it has a solution. Supposing this, let us try to find a suitable generalized Fourier series representation for this solution $u(x,t)$. For reasons not yet clear, let's use the one generated by the Sturm-Liouville problem

$$\phi''(x) = -\lambda\phi \qquad \text{for} \quad 0 < x < L$$

with

$$\phi(0) = 0 \qquad \text{and} \qquad \phi(L) = 0 \quad .$$

Do observe that the boundary conditions at $x = 0$ and $x = L$ are the same as in the above heat flow problem. That is one reason we chose this Sturm-Liouvile problem.

From this Sturm-Liouville problem we get the set of sine functions

$$\left\{ \sin\left(\frac{k\pi}{L}x\right) : k = 1, 2, 3, \dots \right\}$$

which, from our Sturm-Liouville theory, we know is a complete, orthogonal set of functions with respect to weight function $w(x) = 1$. We also know that, for each fixed value of t, we represent the solution to the above heat flow problem, $u(x,t)$, using the corresponding sine series

$$u(x,t) = \sum_{k=1}^{\infty} b_k \sin\left(\frac{k\pi}{L}x\right)$$

[1] Each person reading this should go through the derivation of the heat equation at least once in their life. Reasonable derivations can be found in most introductory texts on partial differential equations.

[2] Part of "solving" many a problem is determining just what the problem is, and what can or should be considered as "known" at the onset. Here, for example, we "know" we can find $u(x,t)$ for all time t. We are wrong.

where

$$b_k = \frac{\left\langle \sin\left(\frac{k\pi}{L}x\right) \,\middle|\, u(x,t) \right\rangle}{\left\| \sin\left(\frac{k\pi}{L}x\right) \right\|^2} = \cdots = \frac{2}{L}\int_0^L u(x,t)\sin\left(\frac{k\pi}{L}x\right)\,dx \quad .$$

Note the that formula for the b_k's depends on t. So these coefficients are not constants but functions of t, and we really should rewrite the above as

$$u(x,t) = \sum_{k=1}^{\infty} b_k(t)\sin\left(\frac{k\pi}{L}x\right) \tag{48.2a}$$

where

$$b_k(t) = \frac{2}{L}\int_0^L u(x,t)\sin\left(\frac{k\pi}{L}x\right)\,dx \quad . \tag{48.2b}$$

Note, also, that the above formula for computing each $b_k(t)$ is pretty much useless since we don't have any formula for $u(x,t)$ — that formula is just what we want to find.

Our goal is to find a formula for $u(x,t)$ by finding the formulas for the b_k's. To do this, we plug the above series representation for u into the heat equation. Naively compute the derivatives in the heat equation by differentiating the terms in the series,

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{\partial}{\partial t}\sum_{k=1}^{\infty} b_k(t)\sin\left(\frac{k\pi}{L}x\right) \\
&= \sum_{k=1}^{\infty} \frac{\partial}{\partial t}\left[b_k(t)\sin\left(\frac{k\pi}{L}x\right)\right] = \sum_{k=1}^{\infty} b_k{}'(t)\sin\left(\frac{k\pi}{L}x\right)
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} &= \frac{\partial^2}{\partial x^2}\sum_{k=1}^{\infty} b_k(t)\sin\left(\frac{k\pi}{L}x\right) \\
&= \sum_{k=1}^{\infty} \frac{\partial^2}{\partial x^2}\left[b_k(t)\sin\left(\frac{k\pi}{L}x\right)\right] \\
&= \sum_{k=1}^{\infty} b_k(t)\left[-\left(\frac{k\pi}{L}\right)^2\sin\left(\frac{k\pi}{L}x\right)\right] = -\sum_{k=1}^{\infty}\left(\frac{k\pi}{L}\right)^2 b_k(t)\sin\left(\frac{k\pi}{L}x\right) \quad .
\end{aligned}$$

With these expressions for the derivatives, equation (48.1a) becomes

$$\sum_{k=1}^{\infty} b_k{}'(t)\sin\left(\frac{k\pi}{L}x\right) + \kappa\sum_{k=1}^{\infty}\left(\frac{k\pi}{L}\right)^2 b_k(t)\sin\left(\frac{k\pi}{L}x\right) = 0 \quad .$$

Letting

$$\mu = \kappa\left(\frac{\pi}{L}\right)^2 \quad ,$$

this can be written more concisely as

$$\sum_{k=1}^{\infty}\left[b_k{}'(t) + k^2\mu b_k(t)\right]\sin\left(\frac{k\pi}{L}x\right) = 0 \quad .$$

Look at this last equation. For each value of t, the left-hand side looks like a sine series which, according to the equation, equals 0 for all x in $(0, L)$. Surely, this is only possible if each coefficient is 0. Here, though, the coefficients are expressions involving the b_k's. So each of these expressions must equal 0. This gives us a bunch of differential equations,

$$\frac{db_k}{dt} + k^2\mu b_k = 0 \qquad \text{for} \quad k = 1, 2, 3, \ldots \quad . \tag{48.3}$$

These differential equations are easy to solve. Each is nothing more than

$$\frac{dy}{dt} + \gamma y = 0$$

with $y = b_k$ and $\gamma = k^2\mu$ — one of the simplest first order linear equations around. You should have no problem confirming that its general solution is $y = Be^{-\gamma t}$ where B is an arbitrary constant. Hence,

$$b_k(t) = B_k\, e^{-k^2\mu t} \qquad \text{for} \quad k = 1, 2, 3, \ldots$$

where the B_k's are yet unknown constants.

With these formulas for the b_k's, formula (48.2a) becomes

$$u(x,t) = \sum_{k=1}^{\infty} B_k\, e^{-k^2\mu t} \sin\left(\frac{k\pi}{L}x\right) \quad . \tag{48.4}$$

In deriving this expression for $u(x,t)$, we assumed $u(x,t)$ exists and satisfies the heat equation (equation (48.1a)) and the endpoint conditions in equation set (48.1b). All that remains is to further refine our expression so it also satisfies the initial condition of equation (48.1c), $u(x,0) = f(x)$. Using the above formula for $u(x,t)$ in this equation, we get

$$f(x) = u(x,0) = \sum_{k=1}^{\infty} B_k\, e^{-k^2\mu\cdot 0} \sin\left(\frac{k\pi}{L}x\right) = \sum_{k=1}^{\infty} B_k \sin\left(\frac{k\pi}{L}x\right)$$

for $0 < x < L$. Cutting out the middle yields

$$f(x) = \sum_{k=1}^{\infty} B_k \sin\left(\frac{k\pi}{L}x\right) \qquad \text{for} \quad 0 < x < L \quad ,$$

which, by an amazing coincidence, looks exactly as if we are representing our known function f by its sine series. Surely then, each B_k must be the corresponding sine coefficient for f,

$$B_k = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{k\pi}{L}x\right) dx \quad .$$

That finishes our derivation. *If* the solution exists and our (occasionally naive) suppositions are valid, then our heat flow problem (equation set (48.1)) is solved by

$$u(x,t) = \sum_{k=1}^{\infty} B_k\, e^{-k^2\mu t} \sin\left(\frac{k\pi}{L}x\right) \tag{48.5a}$$

where

$$\mu = \kappa\left(\frac{\pi}{L}\right)^2 \tag{48.5b}$$

and

$$B_k = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{k\pi}{L}x\right)\,dx \qquad \text{for} \quad k = 1,\ 2,\ 3,\ \ldots \quad . \tag{48.5c}$$

This set of formulas is often called a *formal solution* to the heat equation problem because we obtained it through a process of formal manipulations which seemed reasonable, but were not all rigorously justified.

!►Example 48.1: *Consider solving our heat flow problem when $L = \pi$, $\kappa = \ln 2$, and the rod is initially a constant temperature throughout, say,*

$$f(x) = 100 \quad .$$

Here: $^{\pi}/_{L} = 1$, formula (48.5b) simplifies to

$$\mu = \kappa\left(\frac{\pi}{L}\right)^2 = \ln 2 \quad ,$$

and

$$e^{-k^2\mu t} = e^{-k^2(\ln 2)t} = \left(e^{\ln 2}\right)^{-k^2 t} = 2^{-k^2 t} \qquad \textit{for} \quad k = 1,\ 2,\ 3,\ \ldots \quad .$$

Formula (48.5c) yields

$$\begin{aligned} B_k &= \frac{2}{L}\int_0^L f(x)\sin\left(\frac{k\pi}{L}x\right)\,dx \\ &= \frac{2}{\pi}\int_0^\pi 100\sin(kx)\,dx = \frac{200}{k\pi}\left[1 - (-1)^k\right] \quad . \end{aligned}$$

Hence, according to formula (48.5a), the formal solution to this heat flow problem is

$$\begin{aligned} u(x,t) &= \sum_{k=1}^{\infty} B_k e^{-k^2\mu t}\sin\left(\frac{k\pi}{L}x\right) \\ &= \sum_{k=1}^{\infty} \frac{200}{k\pi}\left[1 - (-1)^k\right] 2^{-k^2 t}\sin(kx) \\ &= \frac{400}{\pi}\left(\frac{1}{2}\right)^{t}\sin(x) + \frac{400}{3\pi}\left(\frac{1}{2}\right)^{3^2 t}\sin(3x) + \frac{400}{5\pi}\left(\frac{1}{2}\right)^{5^2 t}\sin(5x) \\ &\quad + \frac{400}{7\pi}\left(\frac{1}{2}\right)^{7^2 t}\sin(7x) + \frac{400}{9\pi}\left(\frac{1}{2}\right)^{9^2 t}\sin(9x) + \cdots \quad . \end{aligned}$$

Validity and Properties of the Formal Solution

The question remains as to whether formula set (48.5) is a valid solution to our heat flow problem. There are several parts to this question: Does the series converge for all values of x and t of interest? If so, is the resulting function suitably smooth for the expressions in equation set (48.1) to make sense, and if so, does this function satisfy those equations?

Partial answers to these questions, along with some insight, can be gained by examining the terms of our series,

$$B_k\, e^{-k^2 \mu t} \sin\left(\frac{k\pi}{L} x\right) \qquad \text{for} \quad k = 1,\ 2,\ 3,\ \dots \quad .$$

Remember $\mu > 0$ and

$$\begin{aligned} |B_k| \;&=\; \left| \frac{2}{L} \int_0^L f(x) \sin\left(\frac{k\pi}{L}x\right)\, dx \right| \\ &\leq\; \frac{2}{L} \int_0^L |f(x)| \left| \sin\left(\frac{k\pi}{L}x\right)\right| \, dx \;\leq\; \frac{2}{L}\int_0^L |f(x)|\, dx \quad . \end{aligned}$$

So, letting

$$A = \frac{2}{L}\int_0^L |f(x)|\, dx \quad ,$$

we see that

$$\left| B_k\, e^{-k^2 \mu t} \sin\left(\frac{k\pi}{L}x\right)\right| \;\leq\; A\, e^{-k^2 \mu t} \qquad \text{for} \quad k = 1,\ 2,\ 3,\ \dots \quad .$$

If t is also positive, then each $e^{-k^2 \mu t}$ shrinks to 0 very rapidly as $k \to \infty$. This ensures that the series formula for $u(x,t)$ converges absolutely. Consequently, we are assured that $u(x,t)$, as defined by formula set (48.5), is well defined when $0 \leq x \leq L$ and $0 < t$.

"It can be shown" that these exponentially decreasing terms ensure that, as long as $t > 0$, $u(x,t)$ is an infinitely smooth function of both x and t whose partial derivatives can all be computed by differentiating the series term by term. This will allow us to rigorously confirm our formal solution to be a valid solution to our heat flow problem (and a very nice one, at that) when $t > 0$.

On the other hand, if $t < 0$, then $e^{-k^2 \mu t} = e^{k^2 \mu |t|} \to \infty$ as $k \to \infty$. Thus, unless the B_k's shrink to 0 extremely rapidly as $k \to \infty$, the terms of our series solution will blow up, and the series itself diverges whenever $t < 0$.

In short:

> *The series formula given by formula set (48.5) succeeds beautifully as a solution to our heat flow problem for $t > 0$ and, typically, fails miserably for $t < 0$.*

There is something else worth noting about our series solution: Each term in that series,

$$u(x,t) \;=\; \sum_{k=1}^{\infty} B_k\, e^{-k^2 \mu t} \sin\left(\frac{k\pi}{L}x\right) \quad ,$$

rapidly shrinks to 0 as $t \to \infty$. From this it can readily be shown that the maximum and minimum temperatures in the rod must be approaching 0 degrees fairly quickly as t gets large.

?►*Exercise 48.1:* *Let $u(x,t)$ be the infinite series solution found in above example 48.1.*

a: *Verify that*

$$|u(x,t)| \;<\; \frac{400}{\pi} \sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^{tk} \qquad \textit{when} \quad t > 0 \quad . \tag{48.6}$$

b: *Using the above and the formula for computing the sum of a geometric series, show that*

$$|u(x,t)| \ < \ \frac{800}{\pi}\left(\frac{1}{2}\right)^t \qquad \text{when} \quad t \geq 1 \quad .$$

c: *Assuming $u(x,t)$ is the temperature distribution in a rod, what does the above tell you about the maximum temperature in the rod when $t = 1$? when $t = 2$? when $t = 10$?*

?►*Exercise 48.2:* *Again, let $u(x,t)$ be the infinite series solution found in example 48.1, above. This time, consider this infinite series when $t = -1$.*

a: *Write out this series.*

b: *Verify that this series does not converge to anything when $x = {}^{\pi}/_{2}$.*

c: *Show that this series cannot be the sine series for any piecewise continuous function on $(0,\pi)$. (Remember, if it were the sine series for such a function, then the coefficients would be bounded.)*

48.2 The Vibrating String Problem
Setting Up the Problem

Envision an elastic string (such as you might find on any guitar or banjo) stretched between two fixed points on the X–axis, say, from $x = 0$ to $x = L$ (with $L > 0$). For simplicity, we'll assume the string only moves vertically, and we will let

$u(x,t)$ = vertical position at time t of the portion of string located at horizontal position x .

Because the ends of the string are fixed at $x = 0$ and $x = L$, $u(x,t)$ is only defined for $0 \leq x \leq L$, and we have the endpoint conditions

$$u(0,t) \ = \ 0 \qquad \text{and} \qquad u(L,t) \ = \ 0 \quad .$$

After making a few idealizations and applying a little physics, it can be shown that

$$\frac{\partial^2 u}{\partial t^2} \ = \ c^2 \frac{\partial^2 u}{\partial x^2}$$

where c is some positive constant (the reason for using c^2 instead of c will be clear later).[3] This is the basic (one-dimensional) *wave equation*.[4]

We will assume the initial shape of the string is given by the graph of some known function f on $(0,L)$,

$$u(x,0) \ = \ f(x) \qquad \text{for} \quad 0 < x < L \quad .$$

[3] More precisely, $c = \sqrt{{}^{\tau}/_{\rho}}$ where τ and ρ are, respectively, the tension in and the linear density of the string when the stretched string is at rest.

[4] Another famous equation whose derivation we are skipping. Look it up in any decent introductory book on partial differential equations.

For most (unbroken) strings we would expect f to be continuous and piecewise smooth. In addition, since the string is fastened at the endpoints, we should have $f(0) = 0$ and $f(L) = 0$.

As it turns out, this is not quite enough to completely specify $u(x,t)$. An additional initial condition is necessary. We will take that condition to be

$$\left.\frac{\partial u}{\partial t}\right|_{(x,0)} = 0 \qquad \text{for} \quad 0 < x < L \quad .$$

In other words, we assume the string is not moving at time $t = 0$. This would be the case, for example, if we held the string in some fixed shape until releasing it at $t = 0$.

Gathering all the above equations together, we find that $u(x,t)$ must satisfy the following system of equations:

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0 \qquad \text{for} \quad 0 < x < L \tag{48.7a}$$

$$u(0,t) = 0 \qquad \text{and} \qquad u(L,t) = 0 \tag{48.7b}$$

$$u(x,0) = f(x) \qquad \text{for} \quad 0 < x < L \tag{48.7c}$$

$$\left.\frac{\partial u}{\partial t}\right|_{(x,0)} = 0 \qquad \text{for} \quad 0 < x < L \tag{48.7d}$$

Again, there is an implicit requirement that $u(x,t)$ be a sufficiently smooth function for the above equations to make sense. Keep in mind that c is a positive constant and f is a known uniformly continuous and piecewise smooth function on $(0, L)$ satisfying $f(0) = 0 = f(L)$. (Later we will realize that f' must also be piecewise smooth.) While it is reasonable to be interested in the solving this problem just for $t > 0$, such a restriction on t turns out to be mathematically unnecessary. So we will assume the above equations are valid for $-\infty < t < \infty$.

A Formal Solution

The process of finding a solution to our vibrating string problem is very similar to the process we went through to solve our heat flow problem. As then, we begin by supposing a solution $u(x,t)$ exists, and, as with our heat flow problem, the end conditions (equation set (48.7b)) suggest that $u(x,t)$ should be represented by a Fourier sine series on $0 < x < L$ with the coefficients being functions of time,

$$u(x,t) = \sum_{k=1}^{\infty} b_k(t) \sin\left(\frac{k\pi}{L}x\right) \quad . \tag{48.8}$$

As we noted with the heat flow problem, this formula equals 0 when $x = 0$ or $x = L$.

Naively differentiating, we get

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2}{\partial t^2} \sum_{k=1}^{\infty} b_k(t) \sin\left(\frac{k\pi}{L}x\right) = \sum_{k=1}^{\infty} {b_k}''(t) \sin\left(\frac{k\pi}{L}x\right)$$

and

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2}{\partial x^2} \sum_{k=1}^{\infty} b_k(t) \sin\left(\frac{k\pi}{L}x\right) = -\sum_{k=1}^{\infty} b_k(t) \left(\frac{k\pi}{L}\right)^2 \sin\left(\frac{k\pi}{L}x\right) \quad .$$

With these expressions for the derivatives, equation (48.7a) becomes

$$\sum_{k=1}^{\infty} b_k{}''(t) \sin\left(\frac{k\pi}{L}x\right) \;+\; c^2 \sum_{k=1}^{\infty} b_k(t) \left(\frac{k\pi}{L}\right)^2 \sin\left(\frac{k\pi}{L}x\right) \;=\; 0 \quad ,$$

which is written more concisely as

$$\sum_{k=1}^{\infty} \left[b_k{}''(t) \;+\; (k\nu)^2 b_k(t)\right] \sin\left(\frac{k\pi}{L}x\right) \;=\; 0$$

using, for lexicographic convenience,

$$\nu \;=\; \frac{c\pi}{L} \quad .$$

This time each b_k must satisfy the second order linear differential equation

$$\frac{d^2 b_k}{dt^2} \;+\; (k\nu)^2 b_k \;=\; 0 \quad .$$

Again, we have a simple differential equation that should be familiar to anyone who has had an elementary course in differential equations. Its solution is

$$b_k(t) \;=\; A_k \sin(k\nu t) \;+\; B_k \cos(k\nu t)$$

where A_k and B_k are constants yet to be determined.

With this formula for the b_k's , equation (48.8) becomes

$$u(x,t) \;=\; \sum_{k=1}^{\infty} [A_k \sin(k\nu t) \;+\; B_k \cos(k\nu t)] \sin\left(\frac{k\pi}{L}x\right) \quad . \tag{48.9}$$

The A_k's and B_k's will be determined by the initial conditions, equations (48.7c) and (48.7d). For the second initial condition, we will need the partial of u with respect to t , which we might as well (naively) compute here:

$$\begin{aligned} \frac{\partial u}{\partial t} \;&=\; \frac{\partial}{\partial t} \sum_{k=1}^{\infty} [A_k \sin(k\nu t) \;+\; B_k \cos(k\nu t)] \sin\left(\frac{k\pi}{L}x\right) \\ &=\; \sum_{k=1}^{\infty} [A_k k\nu \cos(k\nu t) \;-\; B_k k\nu \sin(k\nu t)] \sin\left(\frac{k\pi}{L}x\right) \quad . \end{aligned} \tag{48.10}$$

Combining formula (48.9) for $u(x,t)$ with the first initial condition gives us

$$\begin{aligned} f(x) \;=\; u(x,0) \;&=\; \sum_{k=1}^{\infty} [A_k \sin(k\nu 0) \;+\; B_k \cos(k\nu 0)] \sin\left(\frac{k\pi}{L}x\right) \\ &=\; \sum_{k=1}^{\infty} [A_k \cdot 0 \;+\; B_k \cdot 1] \sin\left(\frac{k\pi}{L}x\right) \end{aligned}$$

for x in $(0, L)$. Thus, we have

$$f(x) \;=\; \sum_{k=1}^{\infty} B_k \sin\left(\frac{k\pi}{L}x\right) \qquad \text{for} \quad 0 < x < L \quad ,$$

which looks remarkably like an equation we obtained while solving our heat flow problem. As before, we are compelled to conclude that the B_k's are the Fourier sine coefficients for f. That is,

$$B_k = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{k\pi}{L}x\right)dx \qquad \text{for} \quad k = 1, 2, 3, \ldots \quad .$$

The second initial condition, along with formula (48.10), yields

$$\begin{aligned} 0 = \left.\frac{\partial u}{\partial t}\right|_{(x,0)} &= \sum_{k=1}^{\infty}\left[A_k k\nu\cos(k\nu 0) - B_k k\nu\sin(k\nu 0)\right]\sin\left(\frac{k\pi}{L}x\right) \\ &= \sum_{k=1}^{\infty}\left[A_k k\nu\cdot 1 - B_k k\nu\cdot 0\right]\sin\left(\frac{k\pi}{L}x\right) \\ &= \sum_{k=1}^{\infty} A_k k\nu\sin\left(\frac{k\pi}{L}x\right) \qquad \text{for} \quad 0 < x < L \quad , \end{aligned}$$

strongly suggesting that

$$A_k = 0 \qquad \text{for} \quad k = 1, 2, 3, \ldots \quad .$$

Our derivation is complete. If our vibrating string problem (equation set (48.7)) has a solution and our (occasionally naive) computations are valid, then that solution is given by

$$u(x,t) = \sum_{k=1}^{\infty} B_k\cos\left(\frac{kc\pi}{L}t\right)\sin\left(\frac{k\pi}{L}x\right) \tag{48.11a}$$

where

$$B_k = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{k\pi}{L}x\right)dx \qquad \text{for} \quad k = 1, 2, 3, \ldots \quad . \tag{48.11b}$$

Once again, we have derived a "formal solution", that is, a formula obtained through formal (naive) manipulations which we hope can be rigorously verified later.

!► *Example 48.2:* *Consider solving our vibrating string problem assuming $L = 1$ and $c = 3$, and starting with the middle point of the string pulled up a distance of $1/2$ (see figure 48.1). That is, $u(x,0) = f(x)$ with*

$$f(x) = \begin{cases} x & \text{if} \quad 0 \le x \le 1/2 \\ 1 - x & \text{if} \quad 1/2 \le x \le 1 \end{cases} \quad .$$

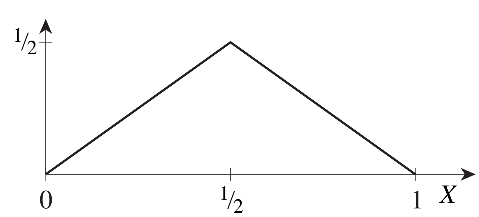

Figure 48.1: The initial shape of the string in example 48.2.

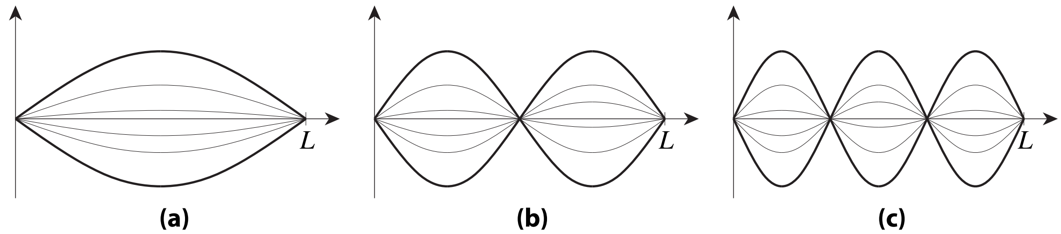

Figure 48.2: The (a) first harmonic, (b) second harmonic, and (c) third harmonic for a vibrating string of length L sketched at various times as functions of x .

With these choices, equation (48.11b) is

$$\begin{aligned} B_k &= \frac{2}{L}\int_0^L f(x)\sin\left(\frac{k\pi}{L}x\right)\,dx \\ &= 2\int_0^{1/2} x\sin(k\pi x)\,dx \;+\; 2\int_{1/2}^{1}(1-x)\sin(k\pi x)\,dx \\ &= \cdots = \sin\left(\frac{k\pi}{2}\right)\left(\frac{2}{k\pi}\right)^2 \quad . \end{aligned}$$

Thus, according to formula (48.11a), the solution to this vibrating string problem is

$$\begin{aligned} u(x,t) &= \sum_{k=1}^{\infty}\sin\left(\frac{k\pi}{2}\right)\left(\frac{2}{k\pi}\right)^2\cos(k3\pi t)\sin(k\pi x) \\ &= 1\left(\frac{2}{\pi}\right)^2\cos(3\pi t)\sin(\pi x) \;+\; 0\left(\frac{2}{2\pi}\right)^2\cos(2\cdot 3\pi t)\sin(2\pi x) \\ &\quad + (-1)\left(\frac{2}{3\pi}\right)^2\cos(3\cdot 3\pi t)\sin(3\pi x) \;+\; 0\left(\frac{2}{4\pi}\right)^2\cos(4\cdot 3\pi t)\sin(4\pi x) \\ &\quad + \cdots \quad . \end{aligned}$$

Harmonics of a Vibrating String

One advantage of the Fourier series solution to our vibrating string problem is that it allows us to analyze the sound produced by such a string by looking at the components of the series solution. For convenience, let's rewrite that solution as

$$u(x,t) = \sum_{k=1}^{\infty} B_k\,u_k(x,t)$$

where

$$u_k(x,t) = \sin\left(\frac{k\pi}{L}x\right)\cos(2\pi\nu_k t) \qquad \text{and} \qquad \nu_k = \frac{kc}{2L} \quad .$$

The individual u_k's are often referred to as the *modes of vibration* or the *harmonics*, with u_1 being the first or "fundamental" mode/harmonic. The graphs of the first three harmonics — $u_1(x,t)$, $u_2(x,t)$, and $u_3(x,t)$ — have been sketched as functions of x for various values of

t in figure 48.2. Notice that $u_k(x,t)$ is nothing more than a sine function of x being scaled by a sinusoid function of time with frequency ν_k. It is that ν_k which determines the pitch of the sound resulting from that mode of vibration. The magnitude of B_k, of course, helps determine the "loudness" of the sound due to the k^{th} harmonic, with the perceived loudness increasing as B_k increases. (However, the relation between B_k and the apparent loudness is not linear and is strongly influenced by the ability of the ear to perceive different pitches.)

In theory, one can produce a "pure tone" corresponding to any one of these frequencies (say ν_3) by imposing just the right initial condition (namely, $u(x,0) = u_3(x,0)$). In practice, this is very difficult, and the sound heard is usually a combination of the sounds corresponding to many of the harmonics. Typically, most of the sound heard is due to the fundamental harmonic, because, typically, people pluck strings in such a manner that the first harmonic is the dominant term in the series solution. For example, whether in a violin or a banjo, ν_1 is approximately 440 cycles/second for a string tuned to A above middle C. The other harmonics provide the "overtones" that modify the sound we hear and help us distinguish between a vibrating violin string and a vibrating banjo string.

!►*Example 48.3:* *In exercise 48.2 we obtained*

$$\begin{aligned} u(x,t) &= 1\left(\frac{2}{\pi}\right)^2 \cos(3\pi t)\sin(\pi x) + 0\left(\frac{2}{2\pi}\right)^2 \cos(2\cdot 3\pi t)\sin(2\pi x) \\ &\quad + (-1)\left(\frac{2}{3\pi}\right)^2 \cos(3\cdot 3\pi t)\sin(3\pi x) + 0\left(\frac{2}{4\pi}\right)^2 \cos(4\cdot 3\pi t)\sin(4\pi x) \\ &\quad + \cdots \end{aligned}$$

as a solution to a vibrating string problem. From this we see that the first four harmonics for this string are

$$u_1(x,t) = \sin(\pi x)\cos(2\pi\nu_1 t) \quad , \quad u_2(x,t) = \sin(2\pi x)\cos(2\pi\nu_2 t) \quad ,$$
$$u_3(x,t) = \sin(3\pi x)\cos(2\pi\nu_3 t) \quad \text{and} \quad u_4(x,t) = \sin(4\pi x)\cos(2\pi\nu_4 t)$$

where

$$\nu_1 = \frac{3}{2} \quad , \quad \nu_2 = 3 \quad , \quad \nu_3 = \frac{9}{2} \quad \text{and} \quad \nu_4 = 6 \quad .$$

The fundamental harmonic frequency is $\nu_1 = {}^3\!/_2$, and the other harmonic frequencies are integral multiples of the fundamental. In this case, the first harmonic is certainly the dominant component of the above solution. However, if the the units on t are seconds, then the first harmonic frequency of ${}^3\!/_2$ cycles per second is somewhat below what most people can hear, and so, as far as most people are concerned, the first harmonic will not contribute significantly to the sound heard from this vibrating string.

Additional Exercises

48.3. *Consider the series solution to the heat flow problem of exercise 48.1 on page 48–5. Using the first 25 terms of this solution, sketch the temperature distribution throughout* *the rod at $t = 0$, $t = 1/10$, $t = 1$, and $t = 10$. (Use the computer math package such as Maple or Mathematica or Mathcad.)*

48.4. *Using the formal solution derived in the first section of this chapter, find the solution to the heat flow problem described in equation set (48.1) on page 48–2 assuming $L = 2$, $\kappa = 3$, and*

a. $f(x) = 5\sin(\pi x)$ **b.** $f(x) = x$

Which of these solutions will be valid for all t and which will just be valid for $t > 0$?

48.5. *If the endpoints of our heat conducting rod are insulated instead of being kept at 0 degrees, then the temperature distribution $u(x,t)$ satisfies the following set of equations:*

$$\frac{\partial u}{\partial t} - \kappa \frac{\partial^2 u}{\partial x^2} = 0 \qquad \text{for} \quad 0 < x < L \,,\ 0 < t$$

$$\left.\frac{\partial u}{\partial x}\right|_{(0,t)} = 0 \qquad \text{and} \qquad \left.\frac{\partial u}{\partial x}\right|_{(L,t)} = 0 \qquad \text{for} \quad 0 < t$$

$$u(x,0) = f(x) \qquad \text{for} \quad 0 < x < L$$

where κ and L are positive constants, and f is piecewise smooth on $(0, L)$.

a. *Why, in this case, would it be better to represent $u(x,t)$ using a cosine series,*

$$u(x,t) = \phi_0(t) + \sum_{k=1}^{\infty} \phi_k(t) \cos\left(\frac{k\pi}{L} x\right) \quad ,$$

instead of the sine series used for the problem in the first section of this chapter? What is the Sturm-Liouville problem that gives rise to this series?

b. *Derive the formal series solution for this heat flow problem.*

c. *Find the solution to this problem assuming $\kappa = 2$, $L = 3$, and*

$$f(x) = \begin{cases} 1 & \text{if} \quad 0 < x < 3/2 \\ 0 & \text{if} \quad 3/2 < x < 3 \end{cases} \quad ,$$

and sketch the temperature distribution (using the first 25 terms of your series solution) for $t = 0$, $t = 1/10$, $t = 1$, and $t = 10$.

d. *What happens to the solution found in the last part as $t \to \infty$? Sketch the temperature distribution "at $t = \infty$".*

e. *What can be said about the differentiability of the solution derived above in the first part of this exercise?*

48.6. *If our heat conducting rod contains sources of heat, and we start with the rod at 0 degrees and keep the endpoints at 0 degrees, then the temperature distribution $u(x,t)$ satisfies the following set of equations:*

$$\frac{\partial u}{\partial t} - \kappa \frac{\partial^2 u}{\partial x^2} = f(x) \qquad \text{for} \quad 0 < x < L \,,\ 0 < t$$

$$u(0,t) = 0 \quad \text{and} \quad u(L,t) = 0 \quad \text{for} \quad 0 < t$$
$$u(x,0) = 0 \quad \text{for} \quad 0 < x < L$$

Again, κ and L are positive constants, and f is piecewise smooth on $(0, L)$.

a. *Derive the formal series solution to this problem assuming that a solution exists. (Hint: Start with formula (48.2a).)*

b. *Find the solution to this problem assuming $\kappa = 4$, $L = 3$, and*

$$f(x) = \begin{cases} 1 & \text{if} \quad 1 < x < 2 \\ 0 & \text{otherwise} \end{cases} \quad ,$$

and sketch the temperature distribution (using the first 25 terms of your series solution) for $t = 0$, $t = 1/10$, $t = 1$, and $t = 10$.

c. *What happens to the solution found in the last part as $t \to \infty$? Sketch the temperature distribution "at $t = \infty$ ".*

d. *What can be said about the differentiability of the solution derived above in the first part of this exercise?*

48.7. *Find the formal solution $u(x,t)$ to the following "vibrating string" problem:*

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0 \quad \text{for} \quad 0 < x < L$$
$$u(0,t) = 0 \quad \text{and} \quad u(L,t) = 0$$
$$u(x,0) = 0 \quad \text{for} \quad 0 < x < L$$
$$\left.\frac{\partial u}{\partial t}\right|_{(x,0)} = f(x) \quad \text{for} \quad 0 < x < L$$

where L and c are positive constants and f is piecewise smooth on $(0, L)$.

48.8. *A more realistic model for the vibrating string that takes into account the dampening of the vibrations due to air resistance is partially given by the equations*

$$\frac{\partial^2 u}{\partial t^2} + 2\beta \frac{\partial u}{\partial t} - c^2 \frac{\partial^2 u}{\partial x^2} = 0 \quad \text{for} \quad 0 < x < L \quad ,$$
$$u(0,t) = 0 \quad \text{and} \quad u(L,t) = 0$$

where L , β and c are positive constants with β being much smaller than c (assume $\beta L < c\pi$ for the following).

a. *Derive, as completely as possible, the formal series solution to the above system of equations. Because no initial conditions are given, your answer should contain arbitrary constants.*

b. *How rapidly do the vibrations die out?*

c. *How does "β" term modify the frequencies at which the individual terms of the solution vibrate?*

Some Answers to Some of the Exercises

WARNING! Most of the following answers were prepared hastily and late at night. They have not been properly proofread! Errors are likely!

1c. $x$max. temp. <128 , max. temp. <64 , max. temp. <0.25

4a. $5e^{-3\pi^2 t}\sin(\pi x)$

4b. $\displaystyle\sum_{k=1}^{\infty}(-1)^{k+1}\frac{4}{k\pi}e^{-3\lambda_k t}\sin\left(\frac{k\pi}{2}x\right)$ where $\lambda_k=\left(\frac{k\pi}{2}\right)^2$

5a. $\phi'' = -\lambda\phi$ with $\phi'(0)=0$ and $\phi'(L)=0$

5b. $\displaystyle A_0 + \sum_{k=1}^{\infty}a_k e^{-\lambda_k t}\cos\left(\frac{k\pi}{L}x\right)$ where $\lambda_k=\kappa\left(\frac{k\pi}{L}\right)^2$,

$$A_0=\frac{1}{L}\int_0^L f(x)\,dx \quad\text{and}\quad a_k=\frac{2}{L}\int_0^L f(x)\cos\left(\frac{k\pi}{L}x\right)\,dx$$

5c. $\displaystyle\frac{1}{2} + \sum_{k=1}^{\infty}\frac{2}{k\pi}\sin\left(\frac{k\pi}{2}\right)e^{-\lambda_k t}\cos\left(\frac{k\pi}{3}x\right)$ where $\lambda_k=2\left(\frac{k\pi}{3}\right)^2$

6a. $\displaystyle\sum_{k=1}^{\infty}\frac{c_k}{\lambda_k}\left[1-e^{-\lambda_k t}\right]\sin\left(\frac{k\pi}{L}x\right)$ where $\lambda_k=\kappa\left(\frac{k\pi}{L}\right)^2$

and $\displaystyle c_k=\frac{2}{L}\int_0^L f(x)\sin\left(\frac{k\pi}{L}x\right)\,dx$

7. $\displaystyle\sum_{k=1}^{\infty}a_k\sin\left(\frac{kc\pi}{L}t\right)\sin\frac{k\pi}{L}x$ where $\displaystyle a_k=\frac{2}{kc\pi}\int_0^L f(x)\sin\frac{k\pi}{L}x\,dx$

8a. $\displaystyle\sum_{k=1}^{\infty}\left[A_k\cos(2\pi\nu_k t)+B_k\sin(2\pi\nu_k t)\right]e^{-\beta t}\sin\left(\frac{k\pi}{L}x\right)$

where the A_k's and B_k's are arbitrary constants and $\nu_k=\frac{1}{L}\sqrt{(kc\pi)^2-\beta^2}$