36

Systems of Differential Equations: General Introduction and Basics

Thus far, we have been dealing with individual differential equations. But there are many applications that lead to sets of differential equations sharing common solutions. In this chapter we will start examining such sets — generally refered to as "systems". In particular, we will develop and describe some of the most basic ideas and terminology, and will derive a select collection of sample systems corresponding to different applications. These systems will later be useful in illustrating the "systems analysis" we will later develop. We will also discover that we can convert most systems of interest (including a "system" involving only one differential equation) into a relatively simple "standard" form. This fact will help shape virtually all of our discussion of systems in the many chapters that follow.

36.1 General Introduction

Basic Terminology and Notions

A k^{th}*-order system of* M *differential equations with* N *unknowns* is simply a collection of M differential equations involving N unknown functions with k being the highest order derivative explicitly appearing in the equations. For brevity, we may refer to a system of M differential equations with N unknowns as a "$M \times N$ system".

!►Example 36.1: *Letting* y_1 , y_2 *and* y_3 *be three unknown functions of* t *, the two differential equations*

$$\frac{d^2y_1}{dt^2} + 3\frac{dy_1}{dt} - \frac{dy_2}{dt} + \sin(t)\,[y_2 - y_1] = y_3$$

and

$$\frac{d^2y_2}{dt^2} - t^2\frac{dy_3}{dt} + y_1 y_2 = 0$$

make up a second-order system of two differential equations with three unknown functions; that is, a second-order 2×3 *system.*

In practice, the number of equations is often equal to the number of unknown functions. Also, in practice, these numbers are rarely stated since it's usually assumed the reader can look at the given system and count the number of equations and unknowns without help.

As suggested by the above example, we will find it convenient to start using t as the variable (so $y' = {}^{dy}/_{dt}$, $y'' = {}^{d^2y}/_{dt^2}$, etc.). This will be the convention for the next several chapters. The symbols denoting the unknown functions in the differential equations, however, will vary. Above, we used y_1 , y_2 and y_3 , but we will find it convenient to use other symbols, including x_1 , x_2 , ... and x_N to represent the unknown functions of t . In particular, when there are only two or three unknown functions, these functions will often be denoted by x , y and, if needed, z ; and if the system arises from some application, then the symbols will often indicate the quantities of interest (as in using v for a velocity).

Now suppose we have a system of M differential equations with unknown functions y_1 , y_2 , ... and y_N , along with some interval of interest I . A (real) *solution* to this system over the interval I is any ordered set of N specific real-valued functions $\hat{y}_1$, $\hat{y}_2$, ... and $\hat{y}_N$ such that all the equations in the system are satisfied for all values of t in the interval I when we let[1]

$$y_1 = \hat{y}_1 \quad , \quad y_2 = \hat{y}_2 \quad , \quad \ldots \quad , \qquad \text{and} \quad y_N = \hat{y}_N \quad .$$

A *general solution* to our system of differential equations (over I) is any ordered set of N formulas describing all possible such solutions. Typically, these formulas include arbitrary constants.[2]

!►Example 36.2: *Consider the relatively simple system*

$$\begin{aligned} x' &= x + 2y \\ y' &= 5x - 2y \end{aligned}$$

over the entire real line (so $I = (-\infty, \infty)$). If we let

$$x(t) = e^{3t} + 2e^{-4t} \qquad \text{and} \qquad y(t) = e^{3t} - 5e^{-4t} \quad ,$$

and plug these formulas for x and y into the first differential equation in our system,

$$x' = x + 2y \quad ,$$

we get

$$\frac{d}{dt}\left[e^{3t} + 2e^{-4t}\right] = \left[e^{3t} + 2e^{-4t}\right] + 2\left[e^{3t} - 5e^{-4t}\right]$$

$$\hookrightarrow \qquad 3e^{3t} - 2 \cdot 4e^{-4t} = [1+2]e^{3t} + [2 - 2 \cdot 5]e^{-4t}$$

$$\hookrightarrow \qquad 3e^{3t} - 8e^{-3t} = 3e^{3t} - 8e^{-3t} \quad ,$$

which is an equation valid for all values of t . So these two functions, x and y , satisfy the first differential equation in the system over $(-\infty, \infty)$.

Likewise, it is easily seen that these two functions also satisfy the second equation:

$$y' = 5x - 2y$$

$$\hookrightarrow \qquad \frac{d}{dt}\left[e^{3t} - 5e^{-4t}\right] = 5\left[e^{3t} + 2e^{-4t}\right] - 2\left[e^{3t} - 5e^{-4t}\right]$$

[1] We could allow the $y_k(t)$'s to be complex valued, but this will not gain us anything with the systems of interest to us, and it would complicate the "graphing" techniques we'll later develop and use.

[2] And it is also typical that the precise interval of interest, I , is not explicitly stated, and may not even be precisely known.

$$\hookrightarrow \qquad 3e^{3t} - 5(-4)e^{-4t} = [5-2]e^{3t} + [5 \cdot 22(-5)]e^{-4t}$$

$$\hookrightarrow \qquad 3e^{3t} + 20e^{-4t} = 3e^{3t} + 20e^{-4t} \quad .$$

Thus, the pair

$$x(t) = e^{3t} + 2e^{-4t} \qquad \text{and} \qquad y(t) = e^{3t} - 5e^{-4t}$$

is a solution to our system (over $(-\infty, \infty)$ *).*

More generally, you can easily verify that, for any choice of constants c_1 *and* c_2 *,*

$$x(t) = c_1 e^{3t} + 2c_2 e^{-4t} \qquad \text{and} \qquad y(t) = c_1 e^{3t} - 5c_2 e^{-4t}$$

satisfies the given system (see exercise 36.4). Later, we will see that this pair of formulas is the general solution for this system. (Note that the two formulas in the general solution in the above example share arbitrary constants. This will be typical.)

On the other hand, plugging the pair

$$x(t) = e^{3t} + 2e^{4t} \qquad \text{and} \qquad y(t) = 2e^{3t} + e^{4t} \quad .$$

into the first equation of our system yields

$$x' = x + 2y$$

$$\hookrightarrow \qquad \frac{d}{dt}\left[e^{3t} + 2e^{4t}\right] = \left[e^{3t} + 2e^{4t}\right] + 2\left[2e^{3t} + e^{4t}\right]$$

$$\hookrightarrow \qquad 3e^{3t} + 8e^{4t} = 5e^{3t} + 4e^{4t}$$

$$\hookrightarrow \qquad 4e^{4t} = 2e^{3t} \quad ,$$

which is not true for every real value t *. So this last pair of functions is not a solution to our system.*

If, in addition to our system of differential equations, we have the values of the solutions and some of their derivatives specified at some single point, then we have an *initial-value problem*, a solution of which is any solution to the system that also satisfies the given initial values. Unsurprisingly, we usually solve initial-value problems by first finding the general solution to the system, and then applying the initial conditions to the general solution to determine the values of the 'arbitrary' constants.

!► *Example 36.3:* *Consider the initial-value problem consisting of the system from the previous example,*

$$\begin{aligned} x' &= x + 2y \\ y' &= 5x - 2y \end{aligned} \quad ,$$

along with the initial conditions

$$x(0) = 0 \qquad \text{and} \qquad y(0) = 1 \quad .$$

In the previous example, it was asserted that

$$x(t) = c_1 e^{3t} + 2c_2 e^{-4t} \quad \text{and} \quad y(t) = c_1 e^{3t} - 5c_2 e^{-4t} \tag{36.1}$$

is a solution to our system for any choice of constants c_1 *and* c_2 *. Using these formulas with the initial conditions, we get*

$$0 = x(0) = c_1 e^{3\cdot 0} + 2c_2 e^{-4\cdot 0} = c_1 + 2c_2$$

and

$$1 = y(0) = c_1 e^{3\cdot 0} - 5c_2 e^{-4\cdot 0} = c_1 - 5c_2 \quad .$$

So, to find c_1 *and* c_2 *, we solve the simple algebraic linear system*

$$\begin{aligned} c_1 + 2c_2 &= 0 \\ c_1 - 5c_2 &= 1 \end{aligned} \quad .$$

Doing so however you wish, you should easily discover that

$$c_1 = \frac{2}{7} \quad \text{and} \quad c_2 = -\frac{1}{7} \quad .$$

which, after plugging these values back into the general formulas for $x(t)$ *and* $y(t)$ *given in equation set (36.1), yields the solution to the given initial-value problem,*

$$x(t) = \frac{2}{7}e^{3t} - \frac{2}{7}e^{-4t} \quad \text{and} \quad y(t) = \frac{2}{7}e^{3t} + \frac{5}{7}e^{-4t} \quad .$$

By the way, you will occasionally hear the term "coupling" describing the extent in which each equation of the system contains different unknown functions. A system is *completely uncoupled* if each equation involves just one of the unknown functions, as in

$$\begin{aligned} x' &= 5x + \sin(x) \\ y'' &= 4y \end{aligned} \quad ,$$

and is *weakly coupled* or *only partially coupled* if at least one of the equations just involves only one unknown functions, as in

$$\begin{aligned} x' &= 5x + 2y \\ y'' &= 4y \end{aligned} \quad .$$

Such systems can be solved in the obvious manner: First solve each equation involving a single unknown function, and then plug those solutions into the other equations, and deal with them.

!►*Example 36.4:* *Consider the system*

$$\begin{aligned} x' &= 5x + 2y \\ y'' &= 4y \end{aligned} \quad .$$

The second equation is the simple linear equation.

$$y'' - 4y = 0$$

whose general solution you can readily find to be

$$y(t) = c_1 e^{2t} + c_2 e^{-2t} \quad .$$

With this, the first equation in the system becomes

$$x' = 5x + 2\left[c_1 e^{2t} + c_2 e^{-2t}\right] \quad ,$$

a first-order linear equation that you should have little trouble solving (see chapter 5). Its general solution is

$$x(t) = -\frac{2}{3}c_1 e^{2t} - \frac{2}{7}c_2 e^{-2t} + c_3 e^{5t} \quad .$$

So, the general solution to our system is

$$x(t) = -\frac{2}{3}c_1 e^{2t} - \frac{2}{7}c_2 e^{-2t} + c_3 e^{5t} \qquad \text{and} \qquad y(t) = c_1 e^{2t} + c_2 e^{-2t} \quad .$$

For the most part, our interest will be in systems that are not weakly coupled.

36.2 A Few Illustrative Applications

Our first example is little more than an observation that the very first example considered in this text naturally led to a system of two equations and two unknowns.

A Falling Object

Way back in section 1.2, we considered object of mass m plummeting towards the ground under the influence of gravity. As we did there, let us set

t = time (in seconds) since the object was dropped

$y(t)$ = vertical distance (in meters) between the object and the ground at time t

$v(t)$ = vertical velocity (in meters/second) of the object at time t

We can view y and v as two unknown functions related by

$$\frac{dy}{dt} = v \quad .$$

Now, in developing a "better model" describing the fall (see the discussion starting on page 12), we took into account air resistance and obtained

$$\frac{dv}{dt} = -9.8 - \kappa v$$

where κ is a positive constant describing how strongly air resistance acts on the falling object. This gives us a system of two differential equations with two unknown functions,

$$\begin{aligned} y' &= v \\ v' &= -9.8 - \kappa v \end{aligned} \quad .$$

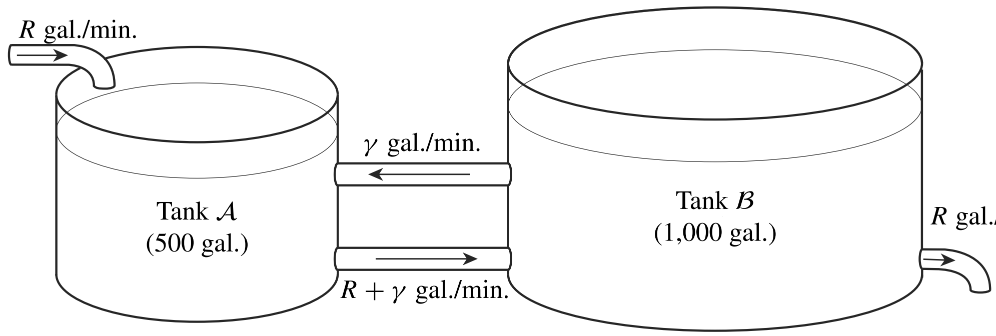

Figure 36.1: A simple system of two tanks containing water/alcohol mixtures.

Fortunately, this is a very weakly coupled system whose second equation is a simple first-order equation involving only the function v. We've already solved it (in example 4.7 on page 91), obtaining

$$v(t) = v_0 + c_1 e^{-\kappa t} \qquad \text{where} \qquad v_0 = -\frac{9.8}{\kappa} \quad .$$

Plugging this back into the first equation of the system yields

$$\frac{dy}{dt} = v_0 + c_1 e^{-\kappa t} \quad ,$$

which we can now integrate:

$$y(t) = \int \frac{dy}{dt}\, dt = \int \left[v_0 + c_1 e^{-\kappa t} \right] dt = v_0 t - \frac{c_1}{\kappa} e^{-\kappa t} + c_2 \quad .$$

So, the general solution to this system is the pair

$$y(t) = v_0 t - \frac{c_1}{\kappa} e^{-\kappa t} + c_2 \qquad \text{and} \qquad v(t) = v_0 + c_1 e^{-\kappa t} \quad .$$

Mixing Problems with Multiple Tanks

Let us expand, slightly, our discussion of "mixing" from section 10.6 on page 256 by considering the situation illustrated in figure 36.1. Here we have two tanks, $\mathcal{A}$ and $\mathcal{B}$. Each minute R gallons of a water/alcohol mix containing C_0 gallons of alcohol per gallon of mix is added to tank $\mathcal{A}$. At the same time, R gallons of the mix in tank $\mathcal{B}$ is drained out. In addition, the two tanks are connected by two pipes, with one pumping liquid from tank $\mathcal{A}$ to tank $\mathcal{B}$ at a rate of $R + \gamma$ gallons per minute, and with the other pumping liquid in the opposite direction, from tank $\mathcal{B}$ to tank $\mathcal{A}$, at a rate of γ gallons per minute.

Following standard conventions, we will let

$t =$ number of minutes since we started the pumping liquid between the two tanks

$x = x(t) =$ gallons of pure alcohol in tank $\mathcal{A}$ at time t

and

$y = y(t) =$ gallons of pure alcohol in tank $\mathcal{B}$ at time t .

Let us assume that tank $\mathcal{A}$ initially contains 500 gallons of pure water, while tank $\mathcal{B}$ initially contains 1,000 gallons of an alcohol-water mix with 90 percent of that mix being alcohol. Note that the input and output flows for each tank cancel out, leaving the total amount of mix in each tank constant. So, our initial conditions are

$$x(0) = 0 \quad \text{and} \quad y(0) = \frac{90}{100} \times 1000 = 900 \quad ,$$

and, at time t, the concentration of alcohol in the two tanks are given, respectively, by

$$\frac{x}{500} \quad \text{and} \quad \frac{y}{1000} \quad .$$

In this system we have four "flows" affecting the rate the amount of alcohol varies in each tank over time, each corresponding to one of the pipes in figure 36.1. In each case the rate at which alcohol is flowing is simply the total flow rate of the mix in the pipe times the concentration of alcohol in that mix.

Thus,

$$\begin{aligned}
\frac{dx}{dt} &= \text{change in the amount of alcohol in tank } \mathcal{A} \text{ per minute} \\
&= \text{rate alcohol is pumped into tank } \mathcal{A} \text{ from the outside} \\
&\quad + \text{rate alcohol is pumped from tank } \mathcal{B} \text{ into tank } \mathcal{A} \\
&\quad - \text{rate alcohol is pumped from tank } \mathcal{A} \text{ into tank } \mathcal{B} \\
&= RC_0 + \gamma \frac{y}{1000} - (R+\gamma)\frac{x}{500} \\
&= RC_0 + \frac{\gamma}{1000} y - \frac{R+\gamma}{500} x \quad ,
\end{aligned}$$

and

$$\begin{aligned}
\frac{dy}{dt} &= \text{change in the amount of alcohol in tank } \mathcal{B} \text{ per minute} \\
&= \text{rate alcohol is pumped from tank } \mathcal{A} \text{ into tank } \mathcal{B} \\
&\quad - \text{rate alcohol is pumped from tank } \mathcal{B} \text{ into tank } \mathcal{A} \\
&\quad - \text{rate alcohol is drained from tank } \mathcal{B} \\
&= (R+\gamma)\frac{x}{500} - \gamma \frac{y}{1000} - R \frac{y}{1000} \\
&= \frac{R+\gamma}{500} x - \frac{R+\gamma}{1000} y \quad ,
\end{aligned}$$

giving us the system

$$\begin{aligned}
x' &= -\frac{R+\gamma}{500} x + \frac{\gamma}{1000} y + RC_0 \\
y' &= \frac{R+\gamma}{500} x - \frac{R+\gamma}{1000} y
\end{aligned} \quad .$$

In particular, if we are adding a 50 percent mixture of alcohol and water at a rate of 3 gallons/minute (so $R = 3$, $C_0 = 1/2$ and $RC_0 = 3/2$, and each minute we have 5 gallons of mix flowing from tank $\mathcal{B}$ to tank $\mathcal{A}$ (so $\gamma = 5$ and $R + \gamma = 8$, then the above system is

$$\begin{aligned}
x' &= -\frac{8}{500} x + \frac{5}{1000} y + \frac{3}{2} \\
y' &= \frac{8}{500} x - \frac{8}{1000} y
\end{aligned} \quad .$$

On the other hand, if we keep $\gamma = 5$ but do not add or drain any mixture (so $R = 0$), then the above system is

$$\begin{aligned} x' &= -\frac{5}{500}x + \frac{5}{1000}y \\ y' &= \frac{5}{500}x - \frac{5}{1000}y \end{aligned} \quad .$$

Foxes in the Rabbit Ranch (A Predator-Prey Model)

In chapter 10, we considered how the number of rabbits in a large ranch varies with time. Let us now assume foxes have entered the fields and, not being vegetarians, have begun dining on the rabbits they can catch, and raising little pups of their own. Our interest is in determining both

$$R(t) = \text{number of rabbits at time } t$$

and

$$F(t) = \text{number of foxes at time } t \quad .$$

For convenience, we'll again take the basic unit of time to be months.

Recall that the basic equation describing the rate of change in the number of rabbits is

$$\frac{dR}{dt} = \beta_R R - \delta_R R$$

where β_R is the monthly birthrate per rabbit, and δ_R is the monthly deathrate per rabbit (that is, δ_R is the fraction of the rabbit population that dies each month). Under ideal conditions — plenty of food and no predators — β_R and δ_R are constants, $\beta_{R,0}$ and $\delta_{R,0}$ (according to the information in chapter 10, $\beta_{R,0} \approx 5/4$ and $\delta_{R,0} \approx 0$). For simplicity, let's take $\delta_{R,0} = 0$ assume there is plenty of food for the rabbits (so $\beta_R = \beta_{R,0}$). But with foxes around, the deathrate can not be assumed constant; it will be a function of the number of foxes,

$$\delta_R = \delta_R(F) \quad .$$

In particular, as the number of foxes increases, so does the deathrate δ_R. With a little thought, you will probably agree that the simplest formula describing a death rate δ_R that increases from the ideal rate of 0 as the number of foxes increases from 0 is

$$\delta_R = \delta_R(F) = \delta_{R,1}F$$

where $\delta_{R,1}$ is some positive constant. Thus,

$$\begin{aligned} \frac{dR}{dt} &= \beta_R R - \delta_R R \\ &= \beta_{R,0}R - \left[\delta_{R,1}F\right]R = \beta_{R,0}R - \delta_{R,1}FR \quad . \end{aligned} \tag{36.2}$$

Likewise, the basic equation describing the rate of change in the number of foxes is

$$\frac{dF}{dt} = \beta_F F - \delta_F F$$

where β_F is the monthly birthrate per fox, and δ_F is the monthly deathrate per fox. In this case, however, both the birthrate and deathrate will depend on the amount of food available for the foxes (i.e., the number of rabbits in the fields). In other words, β_F and δ_F should be treated as functions of R,

$$\beta_F = \beta_F(R) \qquad \text{and} \qquad \delta_F = \delta_F(R) \quad .$$

If $R = 0$, there are no rabbits, and, hence, no food for the foxes, which, in turn, means no new foxes are born, and a large portion of the fox population dies from starvation each month. That is, we should have

$$\beta_F(0) = 0 \qquad \text{and} \qquad \delta_F(0) = \delta_{F,0} \quad .$$

where $\delta_{F,0}$ is the fraction of the fox population that would die from starvation each month if there is no food for them (hence, $\delta_{F,0}$ is some positive number less than or equal to one). As the number of rabbits increases, there is more food and the birthrate will increase while the deathrate will decrease. The simplest formulas describing this are

$$\beta_F(R) = \beta_{F,1}R \qquad \text{and} \qquad \delta_F(R) = \delta_{F,0} - \delta_{F,1}R$$

where $\beta_{F,1}$ and $\delta_{F,1}$ are positive constants. Thus,

$$\begin{aligned} \frac{dF}{dt} &= \beta_F F - \delta_F F \\ &= \left[\beta_{F,1}R\right]F - \left[\delta_{F,0} - \delta_{F,1}R\right]F \\ &= \left[\beta_{F,1} + \delta_{F,1}\right]RF - \delta_{F,0}F \quad . \end{aligned}$$

Combining the last equation with (36.2) (and letting $\gamma_{F,1} = \beta_{F,1} + \delta_{F,1}$) we now have the system

$$\begin{aligned} R' &= \beta_{R,0}R - \delta_{R,1}FR \\ F' &= \gamma_{F,1}RF - \delta_{F,0}F \end{aligned} \tag{36.3}$$

where $\beta_{R,0}$, $\delta_{R,1}$, $\gamma_{F,1}$ and $\delta_{F,0}$ are positive constants that would have to be determined by experiment. In particular, with

$$\beta_{R,0} = {}^{5}\!/\!_{4} \quad , \quad \delta_{R,1} = {}^{1}\!/\!_{16} \quad , \quad \gamma_{F,1} = {}^{1}\!/\!_{300} \quad \text{and} \quad \sigma_{F,0} = {}^{95}\!/\!_{100} \quad ,$$

our system is

$$\begin{aligned} R' &= \frac{5}{4}R - \frac{1}{16}FR \\ F' &= \frac{1}{300}RF - \frac{95}{100}F \end{aligned} \quad .$$

Double Mass-Spring System

Consider the spring system in figure 36.2 with the assumption that there are no frictional forces. Here the "natural length" of each spring — L_1 and L_2, respectively — takes into account the horizontal dimension of the object; that is, if the springs are neither compressed or stretched, then

$$x_1 = L_1 \qquad \text{and} \qquad x_2 - x_1 = L_2 \quad .$$

(If it helps, pretend that each mass is a 'point mass'.)

Now remember, if we have a horizontal spring with spring constant κ and natural length L, then the force exerted by the spring on an object attached to its right end is

$$F_{\text{right}} = -\kappa \times \text{the 'stretch' in the spring} = -\kappa \times [\text{current length of the spring} - L] \quad .$$

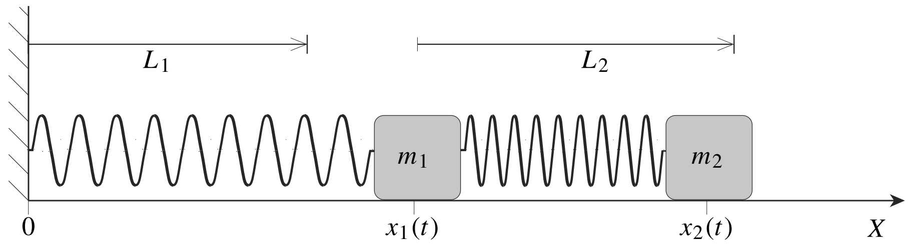

Figure 36.2: A double mass/spring system with objects of mass m_1 and m_2 located at positions $x_1(t)$ and $x_2(t)$, respectively. The first spring, which has a natural length of L_1 and spring constant κ_1, connects the object of mass m_1 to the wall. The second spring, which has a natural length of L_2 and spring constant κ_2, connects the two objects together. In this snapshot, the first spring is stretched, and the second is compressed.

(The negative sign tells us that the force of the spring is in the negative direction if the spring is stretched beyond its natural length, and is positive if the spring is compressed to a length less than its normal length.)

Changing the sign then gives the corresponding force exerted by the spring at the left end,

$$F_{\text{left}} = \kappa \times \text{`stretch' in the spring} = \kappa \times [\text{current length of the spring} - L] \quad .$$

Then applying $F = ma$ to the first object and noting how the "current length" of each spring is computed from $x_1(t)$ and $x_2(t)$, we get

$$\begin{aligned} m_1 \frac{d^2 x_1}{dt^2} &= \text{force of spring 1 on object 1} + \text{force of spring 2 on object 1} \\ &= F_{1,\text{right}} + F_{2,\text{left}} \\ &= -\kappa_1 [x_1 - L_1] + \kappa_2 [(x_2 - x_1) - L_2] \\ &= -[\kappa_1 + \kappa_2] x_1 + \kappa_2 x_2 + [\kappa_1 L_1 - \kappa_2 L_2] \quad . \end{aligned}$$

Since the second object is only attached to the second spring,

$$\begin{aligned} m_2 \frac{d^2 x_2}{dt^2} &= \text{force of spring 2 on object 2} \\ &= F_{2,\text{right}} \\ &= \kappa_2 [L - (x_2 - x_1)] \\ &= \kappa_2 x_1 - \kappa_2 x_2 + \kappa_2 L_2 \quad . \end{aligned}$$

So the motion of the objects in this physical system is described by the solutions to the system

$$\begin{aligned} m_1 x_1{}'' &= -(\kappa_1 + \kappa_2) x_1 + \kappa_2 x_2 + (\kappa_1 L_1 - \kappa_2 L_2) \\ m_2 x_2{}'' &= \kappa_2 x_1 - \kappa_2 x_2 + \kappa_2 L_2 \end{aligned} \quad . \tag{36.4}$$

Equivalently,

$$\begin{aligned} x_1{}'' &= a_{11} x_1 + a_{12} x_2 + b_1 \\ x_2{}'' &= a_{21} x_1 + a_{22} x_2 + b_2 \end{aligned}$$

where

$$a_{11} = -\frac{\kappa_1 + \kappa_2}{m_1} \quad , \quad a_{12} = \frac{\kappa_2}{m_1} \quad , \quad b_1 = \frac{\kappa_1 L_1 - \kappa_2 L_2}{m_1} \quad ,$$

$$a_{21} = \frac{\kappa_2}{m_2} \quad , \quad a_{22} = -\frac{\kappa_2}{m_2} \quad \text{and} \quad b_2 = \frac{\kappa_2 L_2}{m_2} \quad .$$

In particular, suppose the first spring has a natural length of $L_1 = 1$ meter and spring constant of $\kappa_1 = 1$ kg./sec.^{2}, and is attached to an object of mass $m_1 = 1$ kg., while the second spring is shorter and stiffer with natural length $L_2 = 0.2$ meter and spring constant $\kappa_2 = 2.5$ kg./sec.^{2} and is attached on the right to an object of mass $m_2 = 0.1$ kg.. Then (in units of sec.$^{-2}$)

$$a_{11} = -\frac{1 + 2.5}{1} = -\frac{7}{2} \quad , \quad a_{12} = \frac{2.5}{1} = \frac{5}{2} \quad ,$$

$$a_{21} = \frac{2.5}{0.1} = 25 \quad \text{and} \quad a_{22} = -\frac{2.5}{0.1} = -25 \quad ,$$

and (in units of meters·sec.$^{-2}$)

$$b_1 = \frac{1 \cdot 1 - 2.5 \cdot 0.2}{1} = \frac{1}{2} \quad \text{and} \quad b_2 = \frac{2.5 \cdot 0.2}{0.1} = 5 \quad ,$$

and the above system governing the positions of the two objects as functions of time, $x_1(t)$ and $x_2(t)$, is

$$\begin{aligned} x_1'' &= -\tfrac{7}{2}x_1 + \tfrac{5}{2}x_2 + \tfrac{1}{2} \\ x_2'' &= 25x_1 - 25x_2 + 5 \end{aligned} \quad .$$

36.3 Converting High-Order Differential Equations and Systems to Simple First-Order Systems

Converting Single Differential Equations

Let's start with an example.

!►*Example 36.5:* *Consider the second-order differential equation*

$$y'' - 3y' + 8\cos(y) = 0 \quad ,$$

which we will rewrite as

$$y'' = 3y' - 8\cos(y) \quad .$$

Let us introduce two new "unknown" functions y_1 and y_2 related to y by

$$y_1 = y \quad \text{and} \quad y_2 = y' \quad .$$

Then

$$y_1{}' = y' = y_2$$

and

$$y_2{}' = y'' = 3y' - 8\cos(y) = 3y_2 - 8\cos(y_1) \quad .$$

Cutting out the middle of each then gives us the 2×2 *system of first-order differential equations*

$$\begin{aligned} {y_1}' &= y_2 \\ {y_2}' &= 3y_2 - 8\cos(y_1) \end{aligned} \quad .$$

In other words, we have converted the single second-order differential equation

$$y'' - 3y' + 8\cos(y) = 0 \quad .$$

to the above 2×2 *system of first-order differential equations. If we can solve this system, then setting* $y = y_1$ *gives us the solution to the original single second-order differential equation.*

In general, any N^{th}-order differential equation that can be written as

$$y^{(N)} = F\left(t, y, y', y'', \ldots, y^{(N-1)}\right)$$

can be converted to a system of N first-order differential equations with N unknowns by introducing N new unknown functions y_1, y_2, ... and y_N related to y via

$$y_1 = y \quad , \quad y_2 = y' \quad , \quad y_3 = y'' \quad , \quad \ldots \quad \text{and} \quad y_N = y^{(N-1)} \quad .$$

Then we have

$$\begin{aligned} {y_1}' &= y' = y_2 \quad , \\ {y_2}' &= \left(y'\right)' = y'' = y_3 \quad , \\ {y_3}' &= \left(y''\right)' = y''' = y_4 \quad , \\ &\vdots \\ {y_{N-1}}' &= \left(y^{(N-2)}\right)' = y_N \end{aligned}$$

and, finally

$${y_N}' = \left(y^{(N-1)}\right)' = y^{(N)} \quad .$$

But

$$y^{(N)} = F\left(t, y, y', y'', \ldots, y^{(N-1)}\right) = F(t, y_1, y_2, y_3, \ldots, y_N) \quad .$$

So the last equation can be written as

$${y_N}' = F(t, y_1, y_2, y_3, \ldots, y_N)$$

and our original differential equation has been converted to the system of first-order differential equations

$$\begin{aligned} {y_1}' &= y_2 \\ {y_2}' &= y_3 \\ {y_3}' &= y_4 \\ &\vdots \\ {y_{N-1}}' &= y_N \\ {y_N}' &= F(t, y_1, y_2, y_3, \ldots, y_N) \end{aligned} \quad .$$

For convenience, we will call this system the *first-order system corresponding to the original differential equation*. If we can solve this system, then, since $y(t) = y_1(t)$, we automatically have the solution to our original differential equation. And even if we cannot easily solve the system, we will find that some of the tools we'll later develop for first-order systems will greatly aid us in analyzing the possible solutions, especially when the original equation is not linear.

Do note that, using this procedure, any N^{th} order set of initial values

$$y(t_0) = \alpha_1 \quad , \quad y'(t_0) = \alpha_2 \quad , \quad y''(t_0) = \alpha_3 \quad , \quad \ldots \quad \text{and} \quad y^{(n-1)}(t_0) = \alpha_N$$

is converted to values

$$y_1(t_0) = \alpha_1 \quad , \quad y_2(t_0) = \alpha_2 \quad , \quad y_3(t_0) = \alpha_3 \quad , \quad \ldots \quad \text{and} \quad y_N(t_0) = \alpha_N \quad .$$

Let us also note that, as a practical matter, it's not necessary to use subscripted symbols for your new unknowns or to rename your original unknown function. It may be sufficient (and simpler) to just introduce one or two new functions with any convenient names. For example, if y is, say, the height above ground of some falling object, then it is natural to let a second unknown function be the corresponding velocity, $v = y'$ (just as we did in the original falling object example in section 1.2).

On the other hand, the new functions may simply be any convenient letters of the alphabet.

!►*Example 36.6:* *Consider the third-order differential equation*

$$y''' - 3y'' + \sin(y)\,y' = 0 \quad ,$$

which we will rewrite as

$$y''' = 3y'' - \sin(y)\,y' \quad .$$

Introducing the functions x *and* z *related to each other and to* y *by*

$$x = y' \qquad \text{and} \qquad z = y''$$

and observing that

$$x' = y'' = z \qquad , \qquad y' = x$$

and

$$z' = y''' = 3y'' - \sin(y)\,y' = 3z - \sin(y)\,x \quad ,$$

we see that the first-order system of three equations corresponding to our original third-order equation is

$$\begin{aligned} x' &= z \\ y' &= x \\ z' &= 3z - \sin(y)\,x \end{aligned} \qquad .$$

Converting Higher-Order Systems

In a similar manner, we can convert just about any system of differential equations to a larger system of first-order equations.

A single example should suffice.

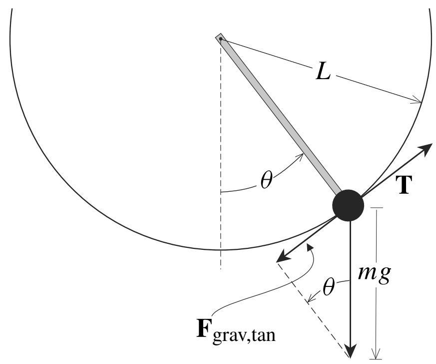

Figure 36.3: The pendulum system with a weight of mass m attached to a massless rod of length L swinging about a pivot point under the influence of gravity.

!►*Example 36.7:* *Consider the system*

$$\begin{aligned} x_1'' &= -\frac{7}{2}x_1 + \frac{5}{2}x_2 + \frac{1}{2} \\ x_2'' &= 25x_1 - 25x_2 + 5 \end{aligned}$$

from our discussion of a double mass-spring system. Let

$$x_3 = x_1' \qquad \text{and} \qquad x_4 = x_2' \quad .$$

Then

$$x_3' = x_1'' \qquad \text{and} \qquad x_4' = x_2'' \quad ,$$

allowing us to rewrite our second-order system of two equations as the first-order system

$$\begin{aligned} x_1' &= x_3 \\ x_3' &= -\frac{7}{2}x_1 + \frac{5}{2}x_2 + \frac{1}{2} \\ x_2' &= x_4 \\ x_4' &= 25x_1 - 25x_2 + 5 \end{aligned} \quad .$$

36.4 The Pendulum

There is one more system that we will want for future use: The system describing the motion of the pendulum illustrated in figure 36.3 consisting of a small weight of mass m attached at one end of a rigid rod, with the other end of the rod attached to a pivot so that the weight can swing around in a circle of radius L in a vertical plane. The forces acting on this pendulum are the downward force of gravity and, possibly, a frictional force from either friction at the pivot point or air resistance.

Let us describe the motion of this pendulum using the angle θ from the vertical downward line through the pivot to the rod, measured (in radians) in the counterclockwise direction. This means that ${}^{d\theta}/_{dt}$ is positive when the pendulum is moving counterclockwise, and is negative when the pendulum is moving clockwise.

Since the motion is circular, and the 'positive' direction is counterclockwise, our interest is in the components of velocity, acceleration and force in the direction of vector $\mathbf{T}$ illustrated figure 36.3. This is the unit vector tangent to the circle of motion pointing in the counterclockwise direction from the current location of the weight. From basic physics and geometry, we know these tangential components of the weight's velocity and acceleration are

$$v_{\text{tan}} = L\frac{d\theta}{dt} \qquad \text{and} \qquad a_{\text{tan}} = L\frac{d^2\theta}{dt^2} \quad .$$

Using basic physics and trigonometry (and figure 36.3), we see that the corresponding component of gravitational force is

$$F_{\text{grav,tan}} = -mg\sin(\theta) \quad .$$

For the frictional force, we'll use what we've used several times before,

$$F_{\text{fric,tan}} = -\gamma v_{\text{tan}} = -\gamma L\frac{d\theta}{dt}$$

where γ is some nonnegative constant — either zero if this is an ideal pendulum having no friction, or a small to large positive value corresponding to a small to large frictional force acting on the pendulum.

Writing out the classic "$ma = F$" equation, we have

$$mL\frac{d^2\theta}{dt^2} = ma_{\text{tan}} = F_{\text{grav,tan}} + F_{\text{fric,tan}} = -mg\sin(\theta) - \gamma L\frac{d\theta}{dt} \quad .$$

Cutting out the middle and dividing through by mL then gives us the slightly simpler equation

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin(\theta) - \kappa\frac{d\theta}{dt} \qquad \text{where} \quad \kappa = \frac{\gamma}{m} \quad . \tag{36.5}$$

Observe that this is a second-order differential equation that, because of the $\sin(\theta)$ term, is nonlinear.

To convert this equation to a first-order system, we will let ω be the angular velocity, ${}^{d\theta}/_{dt}$. So

$$\frac{d\theta}{dt} = \omega$$

and

$$\frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin(\theta) - \kappa\frac{d\theta}{dt} = -\frac{g}{L}\sin(\theta) - \kappa\omega \quad ,$$

giving us the first-order system

$$\begin{aligned} \theta' &= \omega \\ \omega' &= -\frac{g}{L}\sin(\theta) - \kappa\omega \end{aligned} \quad . \tag{36.6}$$

Additional Exercises

36.1. *Consider the following system of differential equations:*

$$\begin{aligned} x' &= 2y \\ y' &= 1 - 2x \end{aligned} \quad .$$

Now determine whether or not each of the following pairs of functions is a solution to the associated system of differential equations.

a. $x(t) = \sin(2t) + \frac{1}{2}$ *and* $y(t) = \cos(2t)$

b. $x(t) = e^{2t} - 1$ *and* $y(t) = e^{2t}$

c. $x(t) = 3\cos(2t) + \frac{1}{2}$ *and* $y(t) = -3\sin(2t)$

36.2. *Consider the following system of differential equations:*

$$\begin{aligned} x' &= 4x - 3y \\ y' &= 6x - 7y \end{aligned} \quad .$$

Now determine whether or not each of the following pairs of functions is a solution to the associated system of differential equations.

a. $x(t) = 6e^{3t}$ *and* $y(t) = 2e^{3t}$

b. $x(t) = 3e^{2t} - e^{-5t}$ *and* $y(t) = 2e^{2t} - 3e^{-5t}$

c. $x(t) = 3e^{2t} + e^{-5t}$ *and* $y(t) = 2e^{2t} - 3e^{-5t}$

36.3. *Consider the following system of differential equations:*

$$\begin{aligned} tx' + 2x &= 15y \\ ty' &= x \end{aligned} \quad .$$

Now determine whether or not each of the following pairs of functions is a solution to the associated system of differential equations.

a. $x(t) = 3t^3$ *and* $y(t) = -t^3$

b. $x(t) = 3t^3$ *and* $y(t) = t^3$

c. $x(t) = -5t^{-5}$ *and* $y(t) = t^{-5}$

36.4. *Verify that, for any choice of constants* c_1 *and* c_2 *, the corresponding pair of functions*

$$x(t) = c_1e^{9t} - c_2e^{-3t} \qquad \text{and} \qquad y(t) = c_1e^{9t} + 2c_2e^{-3t}$$

satisfies the system

$$\begin{aligned} x' &= 5x + 4y \\ y' &= 8x + y \end{aligned} \quad .$$

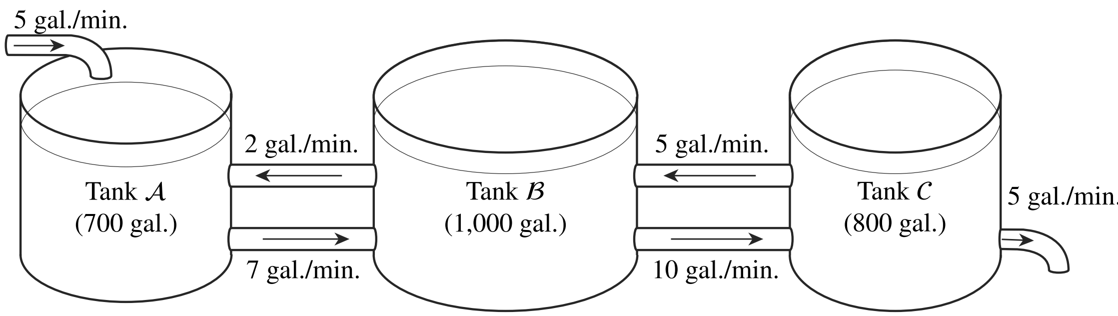

Figure 36.4: The system of three tanks containing water/alcohol mixtures for exercise 36.7. In this scenario, each tank contains a mixture of water and alcohol, and each minute five gallons of mix is added from the upper spigot, with 40 % of that added mix being alcohol.

Then find the solution to this system that satisfies

$$x(0) = 0 \quad \text{and} \quad y(0) = 9 \quad .$$

36.5. *Verify that, for any choice of constants* c_1 *and* c_2 *,*

$$x(t) = c_1 e^{-2t} + 2c_2 e^{5t} \qquad \text{and} \qquad y(t) = -3c_1 e^{-2t} + c_2 e^{5t}$$

satisfies the system

$$\begin{aligned} x' &= 4x + 2y \\ y' &= 3x - y \end{aligned} \quad .$$

Then find the solution to this system that satisfies

$$x(0) = 0 \quad \text{and} \quad y(0) = -21 \quad .$$

36.6. *Solve each of the following weakly coupled systems:*

a. $\begin{aligned} x'' + x &= 0 \\ y' &= x \end{aligned}$ **b.** $\begin{aligned} x' &= 2yx \\ ty' &= y \end{aligned}$ **c.** $\begin{aligned} x' + 2x &= 10z \\ zy' + 5zy &= 15x \\ z' - 3z &= 0 \end{aligned}$

36.7. *Consider the tank system illustrated in figure 36.4. Let* x *,* y *and* z *be, respectively, the amount of alcohol in tanks* $\mathcal{A}$ *,* $\mathcal{B}$ *and* $\mathcal{C}$ *at time* t *(measured in minutes), and find the first-order system of three differential equations describing how* x *,* y *and* z *varies over time.*

36.8. *Consider the mass/spring system illustrated in figure 36.5. Assume there are no frictional forces, and let* κ_j *and* L_j *be, respectively, the spring constant and natural length for the* j^{th} *spring (for* $j = 1, 2,$ *and* 3*).*

a. *Derive the second-order system of two differential equations describing how* x_1 *and* x_2 *vary in time. (As in the derivation of system (36.4) on page 36–10, assume the widths of the two objects are both zero.)*

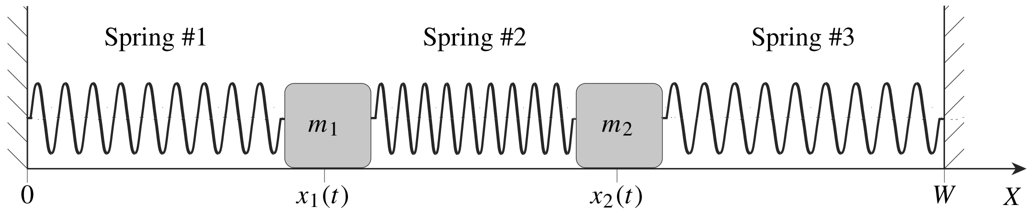

Figure 36.5: The mass/spring system for exercise 36.8 consisting of two objects with masses m_1 and m_2 located at positions $x_1(t)$ and $x_2(t)$, respectively, and attached to each other and to walls at $x = 0$ and $x = W$ by three springs as indicated.

b. *What, in particular, is the system just derived when* $W = 3$ *meters* ,

$$m_1 = m_2 = \frac{1}{2}\,\text{(kilogram)} \quad ,$$

$$L_1 = L_3 = 1\,\text{(meter)} \quad , \quad L_2 = \frac{1}{5}\,\text{(meter)} \quad ,$$

$$\kappa_1 = \kappa_3 = 1\left(\frac{\text{kilogram}}{\text{second}^2}\right) \quad \text{and} \quad \kappa_2 = \frac{5}{2}\left(\frac{\text{kilogram}}{\text{second}^2}\right) \quad ?$$

36.9. *Rewrite the following differential equations as systems of first order equations:*

a. $y'' + 4y' + 2y = 0$

b. $y'' - 8t^2y' - 32y = \sin(t)$

c. $y'' = 4 - y^2$

d. $t^2y'' - 5ty' + 8y = 0$

e. $t^2y'' - ty' + 10y = 0$

f. $4t^2y'' + y = 0$

g. $y'' = 4t^2 - \sin(y')\,y$

h. $y''' + 2y'' - 3y' - 4y = 0$

i. $y''' + y'\left(t^2 + y^2\right) = 0$

j. $y^{(4)} + y^4 = 0$

36.10. *Rewrite each of the following second-order systems as first-order systems:*

a.
$$\begin{aligned} x' - 7y' &= tx^2 \\ y'' + 4y &= 3x \end{aligned}$$

b.
$$\begin{aligned} {x_1}'' + 2x_2{x_1}' + 3x_1{x_2}' &= 0 \\ {x_2}'' - 4{x_2}' + 8x_2 &= (x_1)^2 \end{aligned}$$

c. *The system of two second-order differential equations which is the answer to exercise 36.8 b, above.*

36.11 a. *In section 36.3, we saw that we can convert any second-order differential equation of the form*

$$ay'' + by' + cy = 0$$

to a first order system after introducing a new function x *related to* y *by* $x = y'$. *While this is the "standard" approach, it is not the only approach. In particular, convert each of the following second-order Euler equations to a first-order system by introducing an a new function* x *related to* y *by* $x = ty'$. *(Also, compare the resulting systems to those obtained for the same equations in exercise 36.9, above.)*

i. $t^2y'' - 5ty' + 8y = 0$

ii. $t^2y'' - ty' + 10y = 0$

iii. $4t^2y'' + y = 0$

b. Show that, by introducing a new function x related to y by $x = ty'$, any second-order Euler equation

$$\alpha t^2 y'' + \beta t^2 y' + \gamma y = 0$$

can be converted to the first-order system

$$\begin{aligned} tx' &= \left[1 - \frac{\beta}{\alpha}\right] x - \frac{\gamma}{\alpha} y \\ ty' &= x \end{aligned} .$$

36.12 a. Convert each of the following third-order Euler equations to a first-order system by introducing new functions x and z satisfying $x = ty'$ and $z = tx'$:

i. $t^3 y''' + 2t^2 y'' - 4ty' + 4y = 0$

ii. $t^3 y''' + 4t^2 y'' + 2ty' - 3y = 0$

b. Show that, by introducing new functions x and z satisfying $x = ty'$ and $z = tx'$, any third-order Euler equation

$$\alpha t^3 y''' + \beta t^2 y'' + \gamma t y' + \omega y = 0$$

can be converted to the first-order system

$$\begin{aligned} tx' &= z \\ ty' &= x \\ tz' &= \left(\frac{\beta - \gamma}{\alpha} - 2\right) x - \frac{\omega}{\alpha} y + \left(3 - \frac{\beta}{\alpha}\right) z \end{aligned} .$$

Some Answers to Some of the Exercises

WARNING! Most of the following answers were prepared hastily and late at night. They have not been properly proofread! Errors are likely!

1a. yes, it is
1b. no, it is not
1c. yes, it is
2a. no, it is not
2b. yes, it is
2c. no, it is not
3a. no, it is not
3b. yes, it is
3c. yes, it is
4. $x(t) = 3e^{9t} - 3e^{-3t}$ and $y(t) = 3e^{9t} + 6e^{-3t}$
5. $x(t) = 6e^{-2t} - 6e^{5t}$ and $y(t) = -18e^{-2t} - 3e^{5t}$
6a. $x(t) = c_1\cos(t) + c_2\sin(t)$ and $y(t) = c_1\sin(t) - c_2\cos(t) + c_3$
6b. $x(t) = Ae^{c_1t^2}$ and $y(t) = c_1t$
6c. $x(t) = 2Ae^{3t} + Be^{-2t}$, $y(t) = 6 + \left[\frac{15B}{A}t + C\right]e^{-5t}$ and $z(t) = Ae^{3t}$

7.
$$\begin{aligned} x' &= 2 - \frac{1}{100}x + \frac{1}{500}y \\ y' &= \frac{1}{100}x - \frac{3}{250}y + \frac{1}{160}z \\ z' &= \frac{1}{100}y - \frac{1}{80}z \end{aligned}$$

8a.
$$\begin{aligned} m_1x_1'' &= -(\kappa_1 + \kappa_2)x_1 + \kappa_2x_2 + (\kappa_1L_1 - \kappa_2L_2) \\ m_2x_2'' &= \kappa_2x_1 - (\kappa_2 + \kappa_3)x_2 + \kappa_2L_2 + \kappa_3(W - L_3) \end{aligned}$$

8b.
$$\begin{aligned} x_1'' &= -7x_1 + 5x_2 + 1 \\ x_2'' &= 5x_1 - 7x_2 + 5 \end{aligned}$$

9a.
$$\begin{aligned} x' &= -2y - 4x \\ y' &= x \end{aligned}$$

9b.
$$\begin{aligned} x' &= 32y + 8t^2x + \sin(t) \\ y' &= x \end{aligned}$$

9c.
$$\begin{aligned} x' &= 4 - y^2 \\ y' &= x \end{aligned}$$

9d.
$$\begin{aligned} x' &= 5t^{-1}x - 8t^{-2}y \\ y' &= x \end{aligned}$$

9e.
$$\begin{aligned} x' &= t^{-1}x - 10t^{-2}y \\ y' &= x \end{aligned}$$

9f.
$$\begin{aligned} x' &= -\frac{1}{4t^2}y \\ y' &= x \end{aligned}$$

9g.
$$\begin{aligned} x' &= 4t^2 - \sin(x)\, y \\ y' &= x \end{aligned}$$

9h.
$$\begin{aligned} x' &= z \\ y' &= x \\ z' &= 4y + 3x - 2z \end{aligned}$$

9i.
$$\begin{aligned} x' &= z \\ y' &= x \\ z' &= -x\left(t^2 + y^2\right) \end{aligned}$$

9j.
$$\begin{aligned} {y_1}' &= y_2 \qquad \text{(with } y_1 = y\text{)} \\ {y_2}' &= y_3 \\ {y_3}' &= y_4 \\ {y_4}' &= -(y_1)^4 \end{aligned}$$

10a.
$$\begin{aligned} x' &= 7z + tx^2 \\ y' &= z \\ z' &= -4y + 3x \end{aligned}$$

10b.
$$\begin{aligned} {x_1}' &= x_3 \\ {x_2}' &= x_4 \\ {x_3}' &= -2x_2x_3 - 3x_1x_4 \\ {x_4}' &= 4x_4 - 8x_2 + (x_1)^2 \end{aligned}$$

10c.
$$\begin{aligned} {x_1}' &= x_3 \\ {x_2}' &= x_4 \\ {x_3}' &= -7x_1 + 5x_2 + 1 \\ {x_4}' &= 5x_1 - 7x_2 + 5 \end{aligned}$$

11a i.
$$\begin{aligned} tx' &= 6x - 8y \\ ty' &= x \end{aligned}$$

11a ii.
$$\begin{aligned} tx' &= 2x - 10y \\ ty' &= x \end{aligned}$$

11a iii.
$$\begin{aligned} tx' &= x - \frac{1}{4}y \\ ty' &= x \end{aligned}$$

12a i.
$$\begin{aligned} tx' &= z \\ ty' &= x \\ tz' &= 4x - 4y + z \end{aligned}$$

12a ii.

$$\begin{aligned} tx' &= z \\ ty' &= x \\ tz' &= 3y - z \end{aligned}$$