43

Nonlinear Autonomous Systems of Differential Equations

Let us now turn our attention to nonlinear systems of differential equations. We will not attempt to explicitly solve them — that is usually just too difficult. Instead, we will see that certain things we learned about the trajectories for linear systems with constant coefficients can be applied to sketching trajectories for nonlinear systems. Consequently, we will be drawing pictures describing the qualitative behavior of the solutions. These pictures can be very informative.

Much of the basic theory that we'll develop in the first few sections applies to any "suitably differentiable" $N \times N$ autonomous system of differential equations. However, since we are beginners, we will mainly limit ourselves to 2×2 systems.

43.1 The Systems of Interest and a Little Review

Our interest in this chapter concerns fairly arbitrary 2×2 autonomous systems of differential equations; that is, systems of the form

$$\begin{aligned} x' &= f(x, y) \\ y' &= g(x, y) \end{aligned} \quad ,$$

which we will often write as $\mathbf{x}' = \mathbf{F}(\mathbf{x})$ with the usual understanding that

$$\mathbf{x} = \mathbf{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \quad \text{and} \quad \mathbf{F}(\mathbf{x}) = \begin{bmatrix} f(x, y) \\ g(x, y) \end{bmatrix} \quad .$$

We will assume that our autonomous systems are regular; that is, (as you may recall from chapter 37) we will assume the component functions f and g are continuous and have continuous partial derivatives everywhere on the XY–plane.

Recall that we discussed "trajectories", "direction fields", "phase planes", "critical points and equilibria", and "stability" for such systems in chapter 37. Let's refresh our memories with an example:

!►Example 43.1: *Consider the system*

$$\begin{aligned} x' &= 10x - 5xy \\ y' &= 3y + xy - 3y^2 \end{aligned} \quad . \tag{43.1}$$

To find the critical points, we need to find every ordered pair of real numbers (x, y) *at which both* x' *and* y' *are zero. This means algebraically solving the system*

$$\begin{aligned} 0 &= 10x - 5xy \\ 0 &= 3y + xy - 3y^2 \end{aligned} \quad . \tag{43.2}$$

Fortunately, the first equation factors easily:

$$0 = 10x - 5xy = 5x(2 - y) \quad ,$$

immediately telling us that either $x = 0$ *or* $y = 2$ *.*

If $x = 0$ *, then the second equation in system (43.2) reduces to*

$$0 = 3y + 0 \cdot y - 3y^2 = 3y(1 - y) \quad ,$$

telling us that $y = 0$ *or* $y = 1$ *. This gives us two critical points with* $x = 0$ *:* $(0, 0)$ *and* $(0, 1)$ *.*

On the other hand, if the first equation in system (43.2) holds because $y = 2$ *, then the second equation becomes*

$$0 = 3 \cdot 2 + x \cdot 2 - 3\left(2^2\right) = 2(x - 3) \quad ,$$

implying that $x = 3$ *when* $y = 2$ *. This gives us a third critical point,* $(3, 2)$ *.*

In summary, our system of differential equations has three critical points,

$$(0, 0) \quad , \quad (0, 1) \quad \text{and} \quad (3, 2) \quad .$$

No other choices for (x, y) *will satisfy algebraic system (43.2) (the conditions for a critical point), and any phase portrait for our system of differential equations should include these points (remember these points are the trajectories of the constant or equilibrium solutions to the system).*

A direction field for our system of differential equations, along with a few trajectories, has been sketched in figure 43.1. In that figure, it certainly appears that the critical points $(0, 0)$ *and* $(0, 1)$ *are unstable, and that the critical point* $(3, 2)$ *is asymptotically stable. In fact, from the trajectories and direction arrows in the regions right around the respective points, it even appears that* $(0, 0)$ *is an unstable node,* $(0, 1)$ *is a saddle point, and* $(3, 2)$ *is an asymptotically stable spiral point. We come back to these observations later.*

Some more observations:

1. A constant matrix system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ always has $(0, 0)$ as a critical point, and, if $\mathbf{A}$ is not degenerate (i.e., if $\det(\mathbf{A}) \neq 0$), then $(0, 0)$ is the only critical point. This need not be true for a nonlinear system. As the above example illustrates, we may have several rather different critical points. And it is quite easy to construct systems with no critical points (just use $x' = y^2 + 1$ as one of the equations).

2. If a constant matrix system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ has an asymptotically stable critical point, then every trajectory in the phase plane converges to that critical point. Again, this need not be the case with a nonlinear system. In figure 43.1, it certainly appears that the critical point $(3, 2)$ is asymptotically stable. However, only those trajectories in the first quadrant appear to converge to this point.

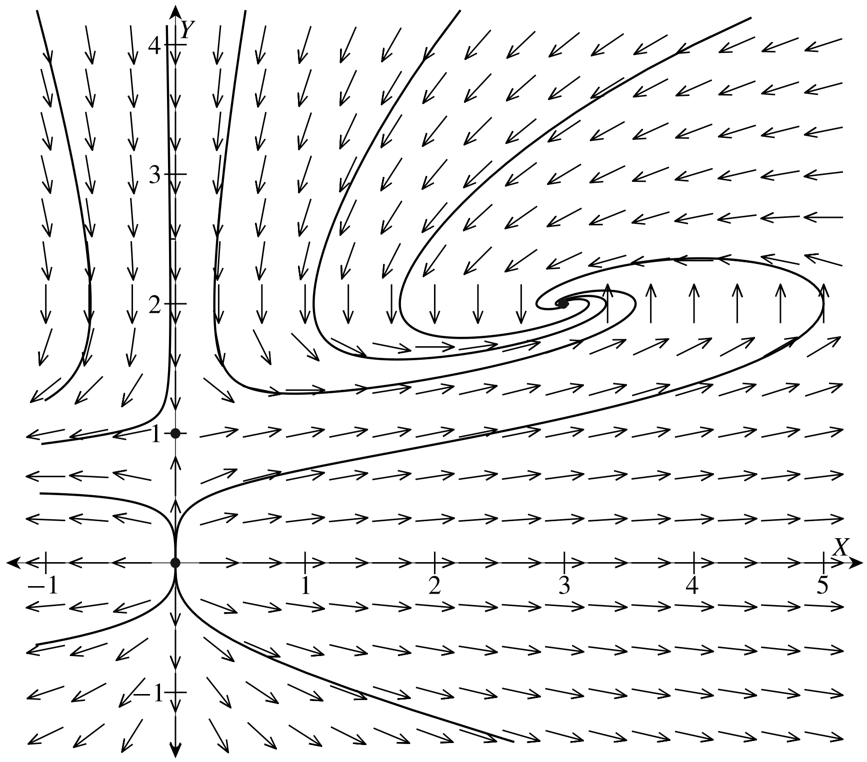


Figure 43.1: A direction field and some trajectories for the system in Example 43.1. This system has critical points $(0, 0)$, $(0, 1)$ and $(3, 2)$

The last observation prompts a little more terminology. We will refer to the region containing of all the trajectories that converge to a given asymptotically stable critical point as either the *region of asymptotic stability* or the *basin of attraction* for that critical point, and a trajectory bounding that region is called a *separatrix* for that region. It figure 43.1, it appears that the first quadrant is the basin of attraction for critical point $(3, 2)$, with any trajectory on the positive X–axis or Y–axis being a separatrix.

43.2 Rewriting Systems Using Jacobian Matrices

The Jacobian Matrix of a System

Associated with the regular system

$$x' = f(x, y)$$
$$y' = g(x, y)$$

is the *Jacobian matrix* of the system, also called the Jacobian matrix of f and g with respect to x and y, or the Jacobian matrix of the vector-valued function $\mathbf{F} = [f, g]^{\mathsf{T}}$. This is the matrix-valued function of x and y, normally denoted by either $\mathbf{J}$ or ${}^{\partial(f,g)}/_{\partial(x,y)}$, given by

$$\mathbf{J}(x, y) = \frac{\partial(f, g)}{\partial(x, y)} = \begin{bmatrix} {}^{\partial f}/_{\partial x} & {}^{\partial f}/_{\partial y} \\ {}^{\partial g}/_{\partial x} & {}^{\partial g}/_{\partial y} \end{bmatrix} \quad .$$

You may have encountered this creature (or its determinant) in other courses involving "two functions of two variables" or "multidimensional change of variables". It will, in a few pages, provide a link between nonlinear and linear systems.

!►*Example 43.2:* *Let's compute the Jacobian matrix for the system in example 43.1,*

$$\begin{aligned} x' &= 10x - 5xy \\ y' &= 3y + xy - 3y^2 \end{aligned} \quad . \tag{43.3}$$

Here,

$$f(x, y) = 10x - 5xy \quad ,$$

$$g(x, y) = 3y + xy - 3y^2 \quad ,$$

and the Jacobian matrix associated with this system is

$$\mathbf{J}(x, y) = \begin{bmatrix} \partial f/\partial x & \partial f/\partial y \\ \partial g/\partial x & \partial g/\partial y \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial}{\partial x}[10x - 5xy] & \frac{\partial}{\partial y}[10x - 5xy] \\ \frac{\partial}{\partial x}\left[3y + xy - 3y^2\right] & \frac{\partial}{\partial y}\left[3y + xy - 3y^2\right] \end{bmatrix} = \begin{bmatrix} 10 - 5y & -5x \\ y & 3 + x - 6y \end{bmatrix} \quad .$$

In particular,

$$\mathbf{J}(1, 3) = \begin{bmatrix} 10 - 5 \cdot 3 & -5 \cdot 1 \\ 3 & 3 + 1 - 6 \cdot 3 \end{bmatrix} = \begin{bmatrix} -5 & -5 \\ 3 & -14 \end{bmatrix} \quad .$$

We will be particularly interested in the Jacobian matrices at the critical points found in the previous exercise. So, let's compute them:

$$\mathbf{J}(0, 0) = \begin{bmatrix} 10 - 5 \cdot 0 & -5 \cdot 0 \\ 0 & 3 + 0 - 6 \cdot 0 \end{bmatrix} = \begin{bmatrix} 10 & 0 \\ 0 & 3 \end{bmatrix} \quad ,$$

$$\mathbf{J}(0, 1) = \begin{bmatrix} 10 - 5 \cdot 1 & -5 \cdot 0 \\ 1 & 3 + 0 - 6(1^2) \cdot 0 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 1 & -3 \end{bmatrix}$$

and

$$\mathbf{J}(3, 2) = \begin{bmatrix} 10 - 5 \cdot 2 & -5 \cdot 3 \\ 2 & 3 + 3 - 6 \cdot 2 \end{bmatrix} = \begin{bmatrix} 0 & -15 \\ 2 & -6 \end{bmatrix} \quad .$$

Recollections of Differentiability

To see the potential value of a Jacobian matrix, we need to review some basic notions regarding "derivatives".

Differential Form for a Function of One Variable

Let us start with a continuous function of one variable $f = f(x)$. Recall that the phrase " $f(x)$ is differentiable at x_0 " means there is a finite number denoted by $f'(x_0)$ given by

$$f'(x_0) = \left.\frac{df}{dx}\right|_{x_0} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \quad .$$

Let $\epsilon(x)$ be the difference between the quotient in the above limit and $f'(x_0)$,

$$\epsilon(x) = \frac{f(x) - f(x_0)}{x - x_0} - f'(x_0) \quad ,$$

and observe both that we can rewrite the last line as

$$f(x) = f(x_0) + \big(f'(x_0) + \epsilon(x)\big)[x - x_0] \quad ,$$

and that, by the definition of $f'(x_0)$ and continuity of f, we must have that ϵ is a continuous function with $\epsilon(x) \to 0$ as $x \to x_0$. In other words, if f is differentiable at x_0, then there is a continuous function $\epsilon(x)$ such that

$$f(x) = f(x_0) + \big(f'(x_0) + \epsilon(x)\big)[x - x_0] \tag{43.4a}$$

with

$$\lim_{x \to x_0} \epsilon(x) = 0 \quad . \tag{43.4b}$$

This is the *differential form* for f about x_0. Note that, if $x \approx x_0$, then $\epsilon(x) \approx 0$, and equation (43.4a) yields the approximation

$$f(x) \approx f(x_0) + f'(x_0)[x - x_0] \qquad \text{when} \quad x \approx x_0 \quad .$$

Differential Form for a Function of Two Variables

Let's now advance to a continuous function of two variables $f = f(x, y)$. Instead of the derivative of f at x_0, we have the partial derivatives f at (x_0, y_0)

$$f_x(x_0, y_0) = \left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} = \lim_{x \to x_0} \frac{f(x, y_0) - f(x_0, y_0)}{x - x_0}$$

and

$$f_y(x_0, y_0) = \left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} = \lim_{y \to y_0} \frac{f(x_0, y) - f(x_0, y_0)}{y - y_0} \quad .$$

It is a little more work, but the general two-dimensional analog to equation set (43.4) can be derived if the partial derivatives f_x and f_y are continuous in a region around (x_0, y_0). What you get, after a little simplification, is that there are continuous functions $\epsilon_1(x, y)$ and $\epsilon_2(x, y)$ such that

$$\begin{aligned} f(x, y) = f(x_0, y_0) &+ \big(f_x(x_0, y_0) + \epsilon_1(x, y)\big)[x - x_0] \\ &+ \big(f_y(x_0, y_0) + \epsilon_2(x, y)\big)[y - y_0] \end{aligned} \tag{43.5a}$$

with

$$\lim_{(x,y) \to (x_0,y_0)} \epsilon_1(x, y) = 0 \qquad \text{and} \qquad \lim_{(x,y) \to (x_0,y_0)} \epsilon_2(x, y) = 0 \quad . \tag{43.5b}$$

Now "for convenience", let

$$A_1 = f_x(x_0, y_0) \quad \text{and} \quad A_2 = f_y(x_0, y_0) \quad ,$$

and observe that equation set (43.5) can be written more concisely as

$$f(x,y) = f(x_0,y_0) + \big[A_1 + \epsilon_1(x,y) \ , \ A_2 + \epsilon_2(x,y)\big] \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix} \tag{43.6a}$$

with

$$\lim_{(x,y)\to(x_0,y_0)} [\epsilon_1(x,y) \, , \, \epsilon_2(x,y)] = [0 \, , \, 0] \quad . \tag{43.6b}$$

From this, we immediately get the approximation

$$f(x,y) \approx f(x_0,y_0) + \big[A_1 \, , A_2\big] \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix} \quad \text{when} \quad (x,y) \approx (x_0,y_0) \quad .$$

By the way, you should have noticed that

$$\big[A_1 \, , A_2\big] = \big[f_x(x_0,y_0) \, , f_y(x_0,y_0)\big]$$

is the *gradient* of $f(x,y)$ at (x_0,y_0), written as a row matrix instead of as a vector. So the gradient should be viewed as the analog of 'the derivative' when dealing with real-valued functions of two variables.

Differential Form for a Vector-Valued Function

Finally, let's consider our vector-valued function

$$\mathbf{F}(\mathbf{x}) = \begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix} \quad .$$

Remember, we are assuming that f, g and the partial derivatives of f and g are continuous. Let (x_0,y_0) be any point in the plane. By the above, we know there are four continuous functions of (x,y) — $\epsilon_{1,1}$, $\epsilon_{1,2}$, $\epsilon_{2,1}$ and $\epsilon_{2,2}$ — which vanish as $(x,y) \to (x_0,y_0)$ and such that

$$f(x,y) = f(x_0,y_0) + \big[A_{1,1} + \epsilon_{1,1}(x,y) \ , \ A_{1,2} + \epsilon_{1,2}(x,y)\big] \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix}$$

and

$$g(x,y) = g(x_0,y_0) + \big[A_{2,1} + \epsilon_{2,1}(x,y) \ , \ A_{2,2} + \epsilon_{2,2}(x,y)\big] \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix} \quad .$$

where

$$A_{1,1} = f_x(x_0,y_0) \quad , \quad A_{1,2} = f_y(x_0,y_0) \quad ,$$
$$A_{2,1} = g_x(x_0,y_0) \quad \text{and} \quad A_{2,2} = g_y(x_0,y_0) \quad .$$

But observe that the above formulas for f and g can be written even more concisely as

$$\begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix} = \begin{bmatrix} f(x_0,y_0) \\ g(x_0,y_0) \end{bmatrix} + \big(\mathbf{A} + \mathbf{E}(\mathbf{x})\big) \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix}$$

where

$$\mathbf{A} = \begin{bmatrix} f_x(x_0, y_0) & f_y(x_0, y_0) \\ g_x(x_0, y_0) & g_y(x_0, y_0) \end{bmatrix} \quad \text{and} \quad \mathbf{E}(\mathbf{x}) = \begin{bmatrix} \epsilon_{1,1}(x, y) & \epsilon_{1,2}(x, y) \\ \epsilon_{2,1}(x, y) & \epsilon_{2,2}(x, y) \end{bmatrix} \quad .$$

Also observe that $\mathbf{A}$ is simply the Jacobian matrix of the system evaluated at (x_0, y_0), $\mathbf{A} = \mathbf{J}(x_0, y_0)$.

What all this means is that we have the following theorem:

Theorem 43.1 (differential form for a vector-valued function of two variables)
Assume $f(x, y)$ and $g(x, y)$ are continuous functions on the XY–plane having continuous partial derivatives everywhere on the plane, and let $\mathbf{F}$ be the corresponding vector-valued function

$$\mathbf{F}(\mathbf{x}) = \begin{bmatrix} f(x, y) \\ g(x, y) \end{bmatrix} \quad \textit{with} \quad \mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} \quad .$$

Then, for each $\mathbf{x}^0 = [x_0, y_0]^\mathsf{T}$ and $\mathbf{x} = [x, y]^\mathsf{T}$,

$$\mathbf{F}(\mathbf{x}) = \mathbf{F}(\mathbf{x}^0) + \big(\mathbf{A} + \mathbf{E}(\mathbf{x})\big)\big[\mathbf{x} - \mathbf{x}^0\big] \tag{43.7a}$$

where

$$\mathbf{A} = \mathbf{J}(x_0, y_0) = \textit{the Jacobian matrix of } \mathbf{F} \textit{ at } (x_0, y_0) \tag{43.7b}$$

and $\mathbf{E}$ is a continuous matrix-valued function of x and y satisfying

$$\mathbf{E}(\mathbf{x}) \to \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \textit{as} \quad \mathbf{x} \to \mathbf{x}^0 \quad . \tag{43.7c}$$

As we will see in the next section, the above theorem has especially important consequences when $\mathbf{x}^0$ is a critical point for the system.

43.3 Linearized Systems and Trajectories Near Critical Points

Let's now restrict our attention to the region near a critical point (x_0, y_0) for our system $\mathbf{x}' = \mathbf{F}(\mathbf{x})$. Then $\mathbf{F}(\mathbf{x}^0) = \mathbf{0}$, and theorem 43.1 immediately yields the following corollary:

Corollary 43.2 (differential form for a nonlinear system)
Let (x_0, y_0) be a critical point for the regular system $\mathbf{x}' = \mathbf{F}(\mathbf{x})$ where

$$\mathbf{F}(\mathbf{x}) = \begin{bmatrix} f(x, y) \\ g(x, y) \end{bmatrix} \quad \textit{and} \quad \mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} \quad .$$

Then, letting $\mathbf{x}^0 = [x_0, y_0]^\mathsf{T}$, the system $\mathbf{x}' = \mathbf{F}(\mathbf{x})$ can be written as

$$\mathbf{x}' = \left(\mathbf{A} + \mathbf{E}(\mathbf{x})\right)\left[\mathbf{x} - \mathbf{x}^0\right] \tag{43.8a}$$

where

$$\mathbf{A} = \mathbf{J}(x_0, y_0) = \text{the Jacobian matrix of } \mathbf{F} \text{ at } (x_0, y_0) \tag{43.8b}$$

and $\mathbf{E}$ *is a continuous matrix-valued function of position satisfying*

$$\mathbf{E}(\mathbf{x}) \rightarrow \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{as} \quad \mathbf{x} \rightarrow \mathbf{x}^0 \quad . \tag{43.8c}$$

For the rest of this section, we will assume that the assumptions in the above corollary hold, and that our system of interest, $\mathbf{x}' = \mathbf{F}(\mathbf{x})$ can be written as described in this corollary. We will also assume that $\mathbf{A}$ is nonsingular. This will ensure that

$$\mathbf{A}[\mathbf{x} - \mathbf{x}^0] \neq \mathbf{0} \qquad \text{whenever} \quad \mathbf{x} \neq \mathbf{x}^0 \quad .$$

Dropping the $\mathbf{E}(\mathbf{x})$ in equation (43.8a) gives us the shifted linear system

$$\mathbf{x}' = \mathbf{A}\left[\mathbf{x} - \mathbf{x}^0\right] \quad ,$$

often referred to as the *linearization of our system about critical point* (x_0, y_0). This is a system we can solve completely (see section 41.2). We can also determine much about the nearby trajectories just from the eigenvalues and eigenvectors for $\mathbf{A}$. Moreover, if $\mathbf{x} = \mathbf{x}(t)$ is a solution to our nonlinear system $\mathbf{x}' = \mathbf{F}(\mathbf{x})$, and we are just looking at a portion of the trajectory near $\mathbf{x}^0$ (where $\mathbf{E}$ is approximately the zero matrix), then

$$\mathbf{x}' = \mathbf{F}(\mathbf{x}) = \left(\mathbf{A} + \mathbf{E}(\mathbf{x})\right)\left[\mathbf{x} - \mathbf{x}^0\right] \approx \mathbf{A}\left[\mathbf{x} - \mathbf{x}^0\right] \quad .$$

But remember, the direction of the direction arrow at each point in a direction field for our system is given by the direction of $\mathbf{x}'$ computed at that point using our system. So, in the region near (x_0, y_0), any direction field of $\mathbf{x}' = \mathbf{F}(\mathbf{x})$ is closely approximated by the direction field of the linearization $\mathbf{x}' = \mathbf{A}[\mathbf{x} - \mathbf{x}^0]$. Hence, in the region near (x_0, y_0), any phase portrait for $\mathbf{x}' = \mathbf{F}(\mathbf{x})$ is closely approximated by corresponding phase portrait for the linearization $\mathbf{x}' = \mathbf{A}[\mathbf{x} - \mathbf{x}^0]$. Moreover, these approximations improve as we look at smaller and smaller regions about the critical point (x_0, y_0).

!►*Example 43.3:* *Again, consider the system*

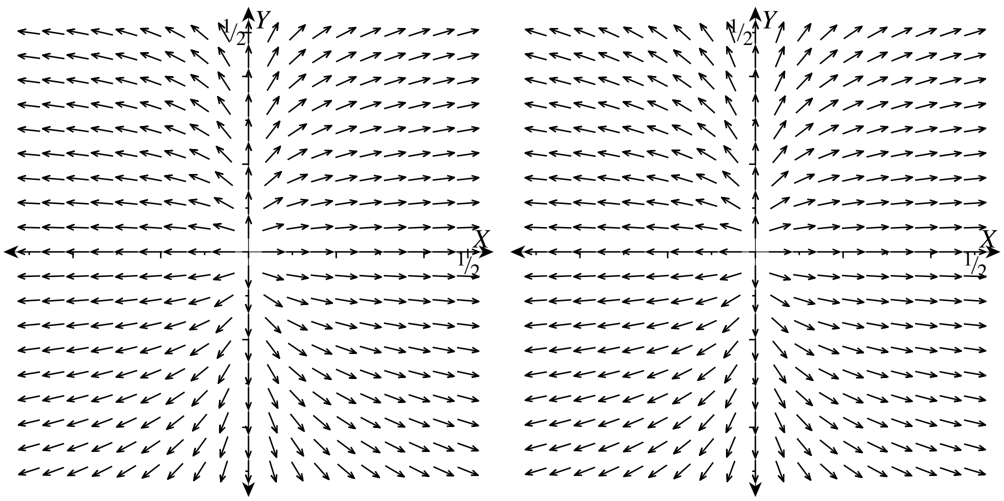
$$\begin{aligned} x' &= 10x - 5xy \\ y' &= 3y + xy - 3y^2 \end{aligned} \quad . \tag{43.9}$$

From examples 43.1 and 43.2, we know $(0, 0)$ *is a critical point for this system, and that the Jacobian matrix of this system at* $(0, 0)$ *is*

$$\mathbf{A} = \mathbf{J}(0, 0) = \begin{bmatrix} 10 & 0 \\ 0 & 3 \end{bmatrix} \quad .$$

So the linearization of our nonlinear system about critical point $(0, 0)$ *is*

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 10 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \quad .$$



(a) **(b)**

Figure 43.2: Direction fields about critical point $(0, 0)$ for **(a)** nonlinear system (43.9) and **(b)** the corresponding linearized system

According to our discussion above, we should expect the direction fields of system (43.9) and the above linearization to be very similar near the critical point $(0, 0)$ *. Just how similar is well illustrated in figure 43.2 in which corresponding direction fields for both have been sketched in a* 1×1 *square about* $(0, 0)$ *.*

Let's go a bit farther and observe that the matrix of the linearized system clearly has eigenpairs

$$\left(3, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \quad \text{and} \quad \left(10, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) \quad ,$$

telling us that the critical point $(0, 0)$ *is an unstable node for the linearized system, with the nonhorizontal trajectories diverging from* $(0, 0)$ *starting out tangent to the vertical axis. And because the direction field of the nonlinear system is so closely approximated by that of the linearized system, it should be clear (especially if we look at the close up views in figure 43.2) that* $(0, 0)$ *must also be an unstable node for our nonlinear system, with most of the trajectories diverging from* $(0, 0)$ *also starting out tangent to the vertical axis. And that was reflected in the phase portrait sketched in figure 43.1.*

As indicated in the above, a careful analysis of the trajectories for our nonlinear system $\mathbf{x}' = \mathbf{F}(\mathbf{x})$ near the critical point (x_0, y_0) starts by rewriting the system as

$$\mathbf{x}' = (\mathbf{A} + \mathbf{E}(\mathbf{x}))[\mathbf{x} - \mathbf{x}^0]$$

or, equivalently, as

$$\mathbf{x}' = \mathbf{A}[\mathbf{x} - \mathbf{x}^0] + \mathbf{E}(\mathbf{x})[\mathbf{x} - \mathbf{x}^0] \quad .$$

We can view $\mathbf{E}(\mathbf{x})[\mathbf{x} - \mathbf{x}^0]$ as an "error term" in using the linearized system to compute $\mathbf{x}'$. Moreover, it is easily verified that this error term is much smaller than the $\mathbf{A}[\mathbf{x} - \mathbf{x}^0]$ term when $\mathbf{x}$ is "sufficiently close" to $\mathbf{x}^0$.[1] Thus, in some region about our critical point, the directions

[1] Remember, we are assuming $\mathbf{A}$ is nonsingular. However, if $\mathbf{A}$ is singular, then it has a zero eigenvalue, and, when

of the direction arrows for $\mathbf{x}' = \mathbf{F}(\mathbf{x})$ are determined primarily by the linearized system with a small adjustment from the error term. From this, we get:

Theorem 43.3 (trajectories about critical points, part I)
Suppose (x_0, y_0) is a critical point of a regular 2×2 autonomous system $\mathbf{x}' = \mathbf{F}(\mathbf{x})$. Let $\mathbf{A}$ be the Jacobian matrix of the system at this critical point, and let r_1 and r_2 be the eigenvalues of $\mathbf{A}$, with $r_1 \le r_2$ if the eigenvalues are real. Then:

1. *If $0 < r_1 < r_2$, then (x_0, y_0) is an unstable node, just as for the linearized system $\mathbf{x}' = \mathbf{A}[\mathbf{x} - \mathbf{x}^0]$. Moreover, all the trajectories diverging from (x_0, y_0) are tangent at this point to the eigenvectors of $\mathbf{A}$, just as for the linearized system.*

2. *If $r_1 < r_2 < 0$, then (x_0, y_0) is an asymptotically stable node, just as for the linearized system $\mathbf{x}' = \mathbf{A}[\mathbf{x} - \mathbf{x}^0]$. Moreover, all the trajectories converging to (x_0, y_0) are tangent at this point to the eigenvectors of $\mathbf{A}$, just as for the linearized system.*

3. *If $r_1 < 0 < r_2$, then (x_0, y_0) is a saddle point, just as for the linearized system $\mathbf{x}' = \mathbf{A}[\mathbf{x} - \mathbf{x}^0]$. Moreover, the trajectories of those solutions that do converge or diverge from the critical point are tangent at the critical point to the corresponding eigenvectors (with those converging to (x_0, y_0) being tangent to the eigenvectors corresponding to r_1, and those diverging from (x_0, y_0) being tangent to the eigenvectors corresponding to r_2. However, most trajectories that pass sufficiently close to (x_0, y_0) turn away from the critical point.*

4. *If the eigenvalues are complex with nonzero real parts, then (x_0, y_0) is a spiral point, just as for the linearized system $\mathbf{x}' = \mathbf{A}[\mathbf{x} - \mathbf{x}^0]$. It is asymptotically stable if the real part is negative, and is unstable if the real part is positive.*

You may have noticed a few cases of interest missing from the above theorem; namely, where the eigenvalues are equal, and where the eigenvalues are purely imaginary. Well:

1. If $0 < r_1 = r_2$ or $r_1 = r_2 < 0$, then the critical point is a star node for the linearized system. However, the error term can add small real and/or imaginary terms to eigenvalues of the matrix $\mathbf{A} + \mathbf{E}(\mathbf{x})$ when $\mathbf{x} \neq \mathbf{x}^0$. This can change the nature of the critical point to either an improper node or a spiral point. Still, the direction arrows will or will not point in the general direction of the critical point, depending on whether the two eigenvalues are negative or positive, respectively.

2. If the eigenvalues are purely imaginary, then the linearized system has a stable center at the critical point. However, the error term could also add a small positive or negative real part to the eigenvalues of matrix $\mathbf{A} + \mathbf{E}(\mathbf{x})$ when $\mathbf{x} \neq \mathbf{x}^0$, changing the elliptical trajectories into spirals either converging to or diverging from the critical point.

Taking the above into consideration leads to our second theorem on trajectories near critical points.

$\mathbf{x} - \mathbf{x}^0$ is a corresponding eigenvector,

$$\mathbf{x}' = \mathbf{A}[\mathbf{x} - \mathbf{x}^0] + \mathbf{E}(\mathbf{x})[\mathbf{x} - \mathbf{x}^0] = \mathbf{0} + \mathbf{E}(\mathbf{x})[\mathbf{x} - \mathbf{x}^0] \quad .$$

Hence, in this case, the error term is not insignificant compared to the term from the linearized system.

Theorem 43.4 (trajectories about critical points, part II)
Suppose (x_0, y_0) *is a critical point of a regular* 2×2 *autonomous system* $\mathbf{x}' = \mathbf{F}(\mathbf{x})$ *. Let* $\mathbf{A}$ *be the Jacobian matrix of the system at this critical point, and let* r_1 *and* r_2 *be the eigenvalues of* $\mathbf{A}$ *. Then:*

1. *If* $0 < r_1 = r_2$ *, then* (x_0, y_0) *is either an unstable node or an unstable spiral point.*
2. *If* $r_1 = r_2 < 0$ *, then* (x_0, y_0) *is either an asymptotically stable node or an asymptotically stable spiral point.*
3. *If the eigenvalues are purely imaginary, then* (x_0, y_0) *can be either a center or a spiral point. Whether it is a stable, asymptotically stable or unstable critical point cannot be determined from just these eigenvalues.*

So let us finish this section by looking at the remaining critical points of the system we've been working on.

!►*Example 43.4:* Once again, consider the nonlinear system

$$\begin{aligned} x' &= 10x - 5xy \\ y' &= 3y + xy - 3y^2 \end{aligned} \quad . \tag{43.10}$$

From examples 43.1 and 43.2, we know this system has three critical points — $(0,0)$, $(0,1)$ and $(3,2)$ — and that the Jacobian matrices of the system at these points are

$$\mathbf{J}(0,0) = \begin{bmatrix} 10 & 0 \\ 0 & 3 \end{bmatrix} \quad , \quad \mathbf{J}(0,1) = \begin{bmatrix} 5 & 0 \\ 1 & -3 \end{bmatrix} \quad \text{and} \quad \mathbf{J}(3,2) = \begin{bmatrix} 0 & -15 \\ 2 & -6 \end{bmatrix} \quad .$$

So, as noted in the last example, the linearized system about critical point $(0,0)$ is

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 10 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \quad .$$

This matrix clearly has eigenpairs $(3, [0,1]^{\mathsf{T}})$ and $(10, [1,0]^{\mathsf{T}})$. Theorem 43.3 assures us that, in fact, $(0,0)$ is an unstable node for the nonlinear system, and that in a region about about $(0,0)$ a phase portrait for the nonlinear system closely is closely approximated by a phase portrait for the linearized system.

Using the Jacobian matrix at $(0,1)$, we get the linearized system about the critical point $(0,1)$,

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x - 0 \\ y - 1 \end{bmatrix} \quad .$$

It is easily verified that the matrix here has eigenpairs

$$\left(-3, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \quad \text{and} \quad \left(5, \begin{bmatrix} 8 \\ 1 \end{bmatrix}\right) \quad ,$$

telling us that this linearized system has a saddle point at $(0,1)$. Hence, so does our nonlinear system (according to theorem 43.3).

Finally, using the Jacobian matrix at $(3, 2)$ *, we get the linearized system about the critical point* $(3, 2)$ *,*

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & -15 \\ 2 & -6 \end{bmatrix} \begin{bmatrix} x-3 \\ y-2 \end{bmatrix} \quad .$$

This eigenvalues of the matrix in this linearization are $r = -3 \pm i\sqrt{21}$ *. So* $(3, 2)$ *is an asymptotically stable spiral point for the linearized system. And theorem 43.3 tells us that this critical point is also an asymptotically stable spiral point for our nonlinear system.*

This verifies the suspicions voiced on page 43–2 after looking at figure 43.1 on page 43–2.

Do observe that we do not actually need to write out the linearization of our system at any given critical point. The important thing is to find the matrix of that system, which is simply the Jacobian matrix of our nonlinear system evaluated at that critical point. All the important information about the trajectories of the nonlinear system about that critical point can then be determined just from that one matrix and the theorems in this section.

43.4 Analyzing Trajectories for Nonlinear Systems

We now have the basic tools for analyzing the behavior of the solutions, and sketching (crude) phase portraits of many a 2×2 nonlinear regular autonomous system $\mathbf{x}' = \mathbf{F}(\mathbf{x})$. We start by first determining the behavior of the trajectories near the critical points via the following procedure:

1. Compute the Jacobian matrix $\mathbf{J}(x, y)$ for the system.

2. Find all the critical points.

3. For each critical point (x_0, y_0) :
   (a) Evaluate the Jacobian matrix at that point, $\mathbf{A} = \mathbf{J}(x_0, y_0)$. This is the matrix for the corresponding linear system at (x_0, y_0) .

   (b) Find the eigenvalues and, if appropriate, the eigenvectors for $\mathbf{A}$.

   (c) Using the eigenvalues and, if appropriate, the eigenvectors of the matrix $\mathbf{A}$ just found, determine (using theorems 43.3 and 43.4) the stability and type of each critical point, and sketch the trajectories in the region near the critical point. Be sure to include indications of the "direction of travel" for them.

Of course, doing the above does require that we can suitably analyze the behavior of the trajectories using the theorems in the last section.

If a more complete phase portrait is desired, then the next step is to fill in the space between the critical points with trajectories sketched in a logical and consistent manner. Show, for example, how trajectories go from one critical point to another, or how they come in from outside the sketched region and either converge to a critical point or leave the sketched region, or … .

The last bit is tricky part. Depending on the system and the "region of interest", try, as well as possible, to determine the general directions of the directions arrows in relevant regions of the sketched phase portrait, and on the edges of region in which the sketch is being made. Construct a minimal direction field to help guide your efforts.

One feature of a phase portrait that can be particularly useful and easy to find are the horizontal and vertical trajectories. They can found by simply finding vertical line segments on which $x' = 0$ or horizontal line segments where $y' = 0$.

!►Example 43.5: *Once again, consider the system*

$$\begin{aligned} x' &= 10x - 5xy \\ y' &= 3y + xy - 3y^2 \end{aligned} \quad . \tag{43.11}$$

On the positive X–axis, $y = 0$ and the above system reduces to

$$\begin{aligned} x' &= 10x > 0 \\ y' &= 0 \end{aligned} \quad . \tag{43.12}$$

which tells us that the direction arrows on the positive X–axis are all parallel to the X–axis and point to the right (as sketched in figure 43.1 on page 43–2). From this (and theorem 37.5 on page 37–20) it follows that the positive X–axis is, itself, a trajectory starting at the origin (where we also have $x' = 0$).

Similarly, on the negative X–axis our system reduces to

$$\begin{aligned} x' &= 10x > 0 \\ y' &= 0 \end{aligned} \quad ,$$

and that means the negative X–axis, oriented away from the origin, is also a trajectory for our system.

(Note that we had to exclude the origin from our computations since the origin, here, is a critical point, and trajectories cannot go through critical points.)

In the above, we used theorem 43.1 to confirm that two horizonal oriented lines were trajectories. Since horizontal and vertical trajectories are relatively common in practice, let us note the following two lemmas (which are immediate corollaries of theorem 43.1):

Lemma 43.5 (horizontal trajectories)
Assume f and g are functions of two variables having continuous partial derivatives everywhere. Assume, further, that there is a horizontal straight-line segment

$$\ell = \{(x, y) : y = y_0 \text{ and } \alpha < x < \beta\}$$

such that, for each (x, y) in ℓ ,

$$f(x, y) \neq 0 \qquad \text{and} \qquad g(x, y) = 0 \quad .$$

Then segment ℓ , properly oriented, is either a trajectory or is contained in a trajectory for the system

$$\begin{aligned} x' &= f(x, y) \\ y' &= g(x, y) \end{aligned} \quad .$$

Lemma 43.6 (vertical trajectories)
Assume f and g are functions of two variables having continuous partial derivatives everywhere. Assume, further, that there is a vertical straight-line segment

$$\ell = \{(x, y) : x = x_0 \text{ and } \alpha < y < \beta\}$$

such that, for each (x, y) in ℓ,

$$f(x, y) = 0 \qquad \text{and} \qquad g(x, y) \neq 0 \quad .$$

Then segment ℓ, properly oriented, is either a trajectory or is contained in a trajectory for the system

$$\begin{aligned} x' &= f(x, y) \\ y' &= g(x, y) \end{aligned} \quad .$$

We will try to illustrate some of the ideas mentioned above in the next two sections.

Again, you may ask why bother with all the above when we can have a computer compute the direction field to begin with. In practice, it's wise to at least find the important points of the phase plane (i.e., the critical points) and to determine the general behavior of the trajectories about these points. This gives you a good idea of the general behavior of the solutions, and a good idea of the regions in the phase plane of particular interest. You can then have the computer construct a direction field (and maybe a few trajectories) in the region of interest to refine your understanding of the trajectories. Moreover, as we will see in the next section, we may be able to carry out the above analysis for a wide class of related systems, obtaining very general (and useful) results for all the systems in this class.

By the way, there is a complication that we have barely touched on: Some of the trajectories may be closed loops. This could certainly occur if a critical point is a center for the corresponding linearized system. It can even arise when none of the critical points are centers. We will deal with systems having "loop trajectories" later, in the next chapter. For now, we will simply avoid such systems.

43.5 Application: Competing Criters (Species)

A Single Species Competing with Itself

Back in chapter 10, we developed two models for population growth. Let us briefly review the "better" model, still assuming our population is a bunch of rabbits in an enclosed field. In that model

$$R(t) = \text{number of rabbits in the field after } t \text{ months}$$

and

$$\frac{dR}{dt} = \text{change in the number of rabbits per month} = \beta R(t)$$

where β is the "net birth rate, ('births − deaths') per rabbit per month". Under ideal conditions, β is a constant β_0, which can be determined from the natural reproductive rate for rabbits and the natural lifetime of a rabbit (see sections 10.2). But assuming β is constant led to a model that predicted an unrealistic number of rabbits in a short time. To take into account the decrease in the net birth rate that occurs when the number of rabbits increases, we added a correction term that decreases the net birth rate as the number of rabbits increases. Using the simplest possible correction term gave us

$$\beta = \beta_0 - \gamma R$$

where γ is some positive constant. Our differential equation for R, $R' = \beta R$, is then

$$\frac{dR}{dt} = (\beta_0 - \gamma R)R \quad .$$

or, equivalently,

$$\frac{dR}{dt} = \gamma(\kappa - R)R \qquad \text{where} \quad \kappa = \text{“the carrying capacity”} = \frac{\beta_0}{\gamma} \quad .$$

This is the "logistic equation", and we discussed it and its solutions in section 10.4. In particular, we discovered that it has a stable equilibrium solution

$$R(t) = \kappa \qquad \text{for all} \quad t \quad .$$

Two Competing Species

Now suppose our field contains both rabbits and gerbils, and they are all eating the same food and competing for the same holes in the ground. Then we should include an additional correction term to the net birth rate β to take into account the additional decrease in net birth rate for the rabbits that occurs as the number of gerbils increases, and the simplest way to add such correction term is to simply subtract some positive constant times the number of gerbils. This gives us

$$\beta = \beta_0 - \gamma R - \alpha G$$

where α is some positive constant and

$$G = G(t) = \text{number of gerbils in the field at time } t \quad .$$

This means that our differential equation for the number of rabbits is

$$\frac{dR}{dt} = (\beta_0 - \gamma R - \alpha G)R \quad .$$

But, of course, there must be a similar differential equation describing the rate at which the gerbil population varies. So we actually have the system

$$\begin{aligned} \frac{dR}{dt} &= (\beta_1 - \gamma_1 R - \alpha_1 G)R \\ \frac{dG}{dt} &= (\beta_2 - \gamma_2 G - \alpha_2 R)G \end{aligned} \tag{43.13}$$

where β_1 and β_2 are the net birth rates per creature under ideal conditions for rabbits and gerbils, respectively, and the γ_k's and α_k's are positive constants that would probably have to be deterimined by experiment and measurement.

Equation set (43.13) is the classic *competing species model*. Let's pick some values for the coefficients and see what the model tells us.

!►Example 43.6: *For our Rabbit/Gerbil system, we'll take*

$$\beta_1 = \frac{5}{4} \quad \text{and} \quad \beta_2 = 3 \quad .$$

The first is from our previous study of rabbit populations in chapter 10, and the second is an uneducated guess. Picking values for the γ_k's and α_k's that seem vaguely reasonable gives us the system

$$\begin{aligned}\frac{dR}{dt} &= \left(\frac{5}{4} - \frac{1}{160}R - \frac{3}{1000}G\right)R \\ \frac{dG}{dt} &= \left(3 - \frac{3}{500}G - \frac{3}{160}R\right)G\end{aligned} \quad . \tag{43.14}$$

The Jacobian matrix of this system is easily computed. It is

$$\mathbf{J}(R,G) = \begin{bmatrix} \frac{5}{4} - \frac{2}{160}R - \frac{3}{1000}G & \frac{-3}{1000}R \\ \frac{-3}{160}G & 3 - \frac{6}{500}G - \frac{3}{160}R \end{bmatrix} \quad .$$

The critical points are the solutions to

$$\begin{aligned}0 &= \left(\frac{5}{4} - \frac{1}{160}R - \frac{3}{1000}G\right)R \\ 0 &= \left(3 - \frac{3}{500}G - \frac{3}{160}R\right)G\end{aligned} \quad . \tag{43.15}$$

The first equation in this algebraic system tells us that either

$$R = 0 \quad \text{or} \quad \frac{1}{160}R + \frac{3}{1000}G = \frac{5}{4} \quad .$$

If $R = 0$, the second equation reduces to

$$0 = \left(3 - \frac{3}{500}G\right)G$$

which means that either

$$G = 0 \quad \text{or} \quad G = 500 \quad .$$

So two critical points are $(R,G) = (0,0)$ and $(R,G) = (0,500)$.

If, on the other hand, the first equation in algebraic system (43.15) holds because

$$\frac{1}{160}R + \frac{3}{1000}G = \frac{5}{4} \quad .$$

Then the system's second equation can only hold if either

$$G = 0 \quad \text{or} \quad \frac{3}{500}G + \frac{3}{160}R = 3 \quad .$$

If $G = 0$, then we can solve the first equation in the system, obtaining

$$R = \frac{5}{4} \cdot 160 = 200 \quad .$$

So $(R, G) = (200, 0)$ *is one critical point. Looking at what remains, we see that there is one more critical point, and it satisfies the simple algebraic linear system*

$$\begin{aligned} \frac{1}{160}R + \frac{3}{1000}G &= \frac{5}{4} \\ \frac{3}{160}R + \frac{3}{500}G &= 3 \end{aligned} \quad .$$

You can easily verify that the solution to this is $(R, G) = (80, 250)$ *.*

So the critical points for our system are (R, G) *equaling*

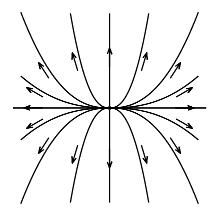
$$(0, 0) \quad , \quad (0, 500) \quad , \quad (200, 0) \qquad \text{and} \qquad (80, 250) \quad .$$

Now let's look at each of these points:

1. $\boldsymbol{(R, G) = (0, 0)}$**:** *Plugging* $(R, G) = (0, 0)$ *into the Jacobian matrix yields*

$$\mathbf{J}(0, 0) = \begin{bmatrix} \frac{5}{4} & 0 \\ 0 & 3 \end{bmatrix} \quad ,$$

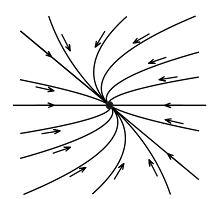
which clearly has eigenvalues $^5/_4$ *and* 3 *, with corresponding eigenvectors* $[1, 0]^\mathsf{T}$ *and* $[0, 1]^\mathsf{T}$ *, respectively. Thus, this critical point is an unstable node, and a phase portrait about this point will be similar to the sketch at the right.*



2. $\boldsymbol{(R, G) = (200, 0)}$**:** *Plugging* $(R, G) = (200, 0)$ *into the Jacobian matrix yields*

$$\mathbf{J}(200, 0) = \begin{bmatrix} \frac{5}{4} - \frac{2}{160} \cdot 200 & \frac{-3}{1000} \cdot 200 \\ 0 & 3 - \frac{3}{160} \cdot 200 \end{bmatrix} = \begin{bmatrix} -\frac{5}{4} & \frac{-3}{5} \\ 0 & -\frac{3}{4} \end{bmatrix} \quad ,$$

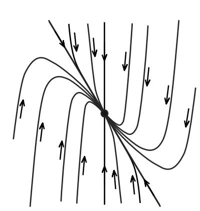
which you can easily verify has eigenvalues $-^5/_4$ *and* $-^3/_4$ *, with corresponding eigenvectors* $[1, 0]^\mathsf{T}$ *and* $[6, -5]^\mathsf{T}$ *, respectively. Thus, this critical point is an asymptotically stable node, and a phase portrait about this point will be similar to the sketch at the right.*



3. $\boldsymbol{(R, G) = (0, 500)}$**:** *Plugging* $(R, G) = (0, 500)$ *into the Jacobian matrix yields*

$$\mathbf{J}(0, 500) = \begin{bmatrix} \frac{5}{4} - \frac{3}{1000} \cdot 500 & 0 \\ \frac{-3}{160} \cdot 500 & 3 - \frac{6}{500} \cdot 500 \end{bmatrix} = \begin{bmatrix} -\frac{1}{4} & 0 \\ -\frac{75}{8} & -3 \end{bmatrix} \quad ,$$

which you can easily verify has eigenvalues $-^1/_4$ *and* -3 *, with corresponding eigenvectors* $[22, -75]^\mathsf{T}$ *and* $[0, 1]^\mathsf{T}$ *, respectively. Thus, this critical point is also an asymptotically stable node, and a phase portrait about this point will be similar to the sketch at the right.*



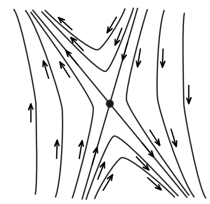
4. **$(R, G) = (80, 250)$:** *Plugging $(R, G) = (80, 250)$ into the Jacobian matrix yields*

$$\mathbf{J}(80, 250) = \cdots = \begin{bmatrix} -\frac{1}{2} & -\frac{6}{25} \\ -\frac{75}{16} & -\frac{3}{2} \end{bmatrix} .$$

You should have little difficulty in verifying that

$$\left(1 - \sqrt{7}, \begin{bmatrix} 12 \\ -75 + 50\sqrt{7} \end{bmatrix}\right) \quad \text{and} \quad \left(1 + \sqrt{7}, \begin{bmatrix} 12 \\ -75 - 50\sqrt{7} \end{bmatrix}\right) ,$$

are eigenpairs for this matrix. Note that $1 - \sqrt{7} < 0 < 1 + \sqrt{7}$. Thus, the critical point $(80, 250)$ is a saddle point, and a phase portrait about it will be similar to that sketched to the right.



Now that we have the critical points and know something of the trajectories near these points, let's plot these critical points and, in a small region about each critical point, sketch simplified versions of the phase portraits of the corresponding linearized systems. This yields figure 43.3a.

To fill in the rest of our phase portrait, it helps observe that our system

$$\frac{dR}{dt} = \left(\frac{5}{4} - \frac{1}{160}R - \frac{3}{1000}G\right) R$$
$$\frac{dG}{dt} = \left(3 - \frac{3}{500}G - \frac{3}{160}R\right) G$$

can be rewritten as

$$\begin{aligned} \frac{dR}{dt} &= \left(\frac{1}{160}[200 - R] - \frac{3}{1000}G\right) R \\ \frac{dG}{dt} &= \left(\frac{3}{500}[500 - G] - \frac{3}{160}R\right) G \end{aligned} \qquad . \qquad (43.16)$$

Note that the values 200 and 500 in the above system are, respectively, the R and G values of the critical points on the R–axis and G–axis. Note, also, that this system simplifies greatly when either $G = 0$ or $R = 0$.

If $G = 0$, the above system reduces to

$$\begin{aligned} \frac{dR}{dt} &= \frac{1}{160}[200 - R]R \\ \frac{dG}{dt} &= 0 \end{aligned} \qquad . \qquad (43.17)$$

Hence, the direction arrow at each non-critical point $(R, 0)$ of the R–axis is parallel to the R–axis (as sketched in figure 43.3a). In particular, when $0 < R < 200$, then $R' > 0$ and the arrow points to the right. And when $200 < R$, then $R' < 0$ and the arrow points to the left. This (along with lemma 43.5 on page 43–13) tells us that there is one trajectory along the positive R–axis from the origin to the critical point $(200, 0)$, and another trajectory towards

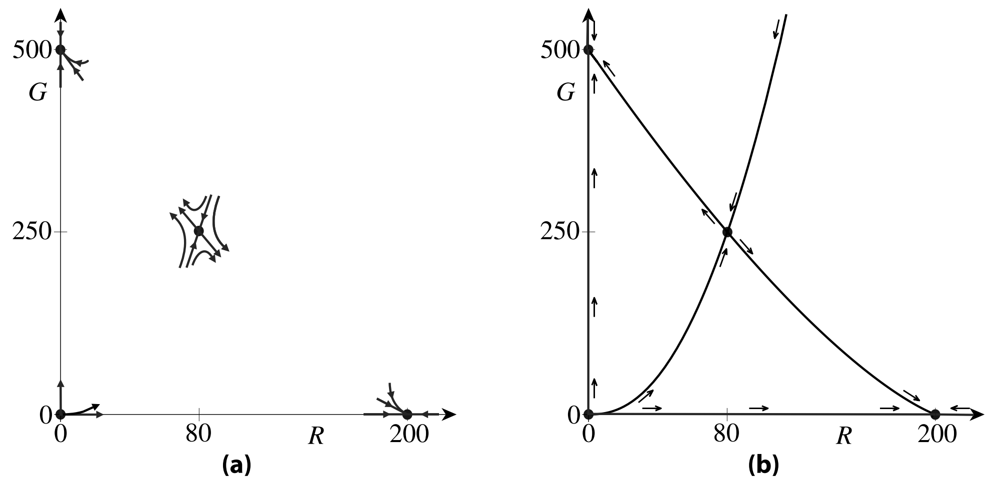


Figure 43.3: Constructing a phase portrait for the rabbit/gerbil system of example 43.6: **(a)** The critical points with portions of nearby trajectories. **(b)** A minimal phase portrait.

the critical point along the rest of the positive R*–axis. Knowing this, we can now sketch these two trajectories, as done in figure 43.3b.*

Likewise, when $R = 0$ *, system (43.16) reduces to*

$$\begin{aligned}\frac{dR}{dt} &= 0 \\ \frac{dG}{dt} &= \frac{3}{500}[500 - G]G\end{aligned} \quad ,$$

telling us that the direction arrow at each noncritical point $(0, G)$ *on the positive* G*–axis is parallel to the* G*–axis and pointing towards the critical point* $(0, 500)$ *. From this (and lemma 43.6), we see that there is one trajectory along the* G*–axis from the origin to this critical point, and another trajectory directed towards this critical point along the rest of the positive* G*–axis. Naturally, we add these trajectories to our sketch, as done in 43.3b.*

The fact that the trajectory through any non-critical point on the positive R*–axis and* G*–axis remains on the respective axis tells us that no trajectory crosses either the positive* R*–axis or the positive* G*–axis. Thus, any trajectory passing through a point* (R, G) *with* $R \geq 0$ *and* $G \geq 0$ *is totally contained in the quarter-plane with* $R \geq 0$ *and* $G \geq 0$ *. This assures us of two things:*

1. *The model is realistic in that it never predicts a negative number of rabbits or gerbils provided we start with a nonnegative numbers of rabbits and gerbils.*

2. *We can restrict our attention to the first quadrant and its boundary.*

Since we cannot actually sketch a phase portrait over the entire first quadrant, let us choose our "area of interest" to be a rectangle containing the critical points, and bounded below and to the left by the R*–axis and* G*–axis.*

What about the trajectories passing through the edges of this region other than the two axes? Well, to begin with, suppose (R_0, G_0) *is any point with* $R_0 \geq 200$ *and* $G_0 > 0$ *. Then, at this point,*

$$\frac{dR}{dt} = \left(\frac{1}{160}[200 - R_0] - \frac{3}{1000}G_0\right) R_0 < 0 \quad .$$

That is, the horizontal component of the direction arrow at this point is negative. Consequently, any trajectory in the upper half plane intersecting a vertical line to the right of the critical point $(200, 0)$ *must be crossing that line with a direction of travel towards the left.*

Similarly, at any point (R_0, G_0) *with* $R_0 > 0$ *and* $G_0 \geq 500$,

$$\frac{dG}{dt} = \left(\frac{3}{500}[500 - G_0] - \frac{3}{160}R_0\right) G_0 < 0 \quad ,$$

and, from this, it follows that any trajectory in the first octant intersecting a horizontal line above the critical point $(0, 500)$ *must be crossing that line with a direction of travel in a downwards direction.*

What all this tells us is that every trajectory passing through the upper or righthand boundary of our region of interest must be directed into the region.

Next, let's attempt some complete trajectories off of the axes.

In figure 43.3a we see that there are two trajectories "leaving" critical point $(80, 250)$. *Let us (somewhat naively) attempt to extend these trajectories, starting with the one initially heading "down and to the right". Because of what we now know about the trajectories, this trajectory cannot head out of our region of interest, nor should we expect it to meander aimlessly in the region. A reasonable expectation is that it heads towards one of the critical points other than the unstable node at* $(0, 0)$. *Let us keep things as simple as possible and naively continue extending this trajectory "down and to the right" until it ends at the stable node* $(200, 0)$ *, as done in figure 43.3b.*

Likewise, let us naively extend the trajectory "leaving" critical point $(80, 250)$ *and initially heading "up and to the left" to the stable node which is "up and to the left", namely, the point* $(0, 500)$ *, as also indicated in figure 43.3b.*

(The critical reader would rightfully be concerned at how we chose the end points of these two trajectories. That reader is encouraged to attempt exercise 43.6 on page 43–31 to better justify the naive assumptions made above.)

Now, consider the two trajectories that "end" at $(80, 250)$ *, according to the sketch in figure 43.3a. Clearly, the only point at which the one coming in from below could have started is the unstable node* $(0, 0)$. *So let's extend this trajectory back to* $(0, 0)$ *, remembering to have it "leave" this node tangent to the* R*–axis. This leaves the trajectory coming into* $(80, 250)$ *from above, and since there are no other critical points from which this trajectory can begin, it seems reasonable that it must be one of the trajectories "coming into the region". So let's draw it as such.*

The result is the minimal phase portrait in figure 43.3b.

To finish our phase portrait, we simply add a few trajectories starting at $(0, 0)$ *(the only unstable node) or coming in from above or to the right of the region of interest, and converging to whichever stable node is possible. Remember to take into account the fact that* $(80, 250)$ *is a saddle point, and the fact that the trajectories become tangent to certain lines as the trajectories approach the stable nodes. The end result should be similar to that sketched in figure 43.4a.*

So, what can we conclude from our final phase portrait in figure 43.4a? From that phase portrait, it is clear (assuming our model and rough sketch of the phase portrait is reasonably accurate) that both species can coexist forever with 80 rabbits and 250 gerbils. However, this equilibrium state is very unstable. It is much more likely that one or the other species will die out, leaving us with a population of either 200 rabbits and no gerbils, or no rabbits and 500 gerbils.

(For comparison, a more accurately phase portrait generated by a computer has been sketched in figure 43.4b. Note that we would obtain the same conclusions from it as we drew

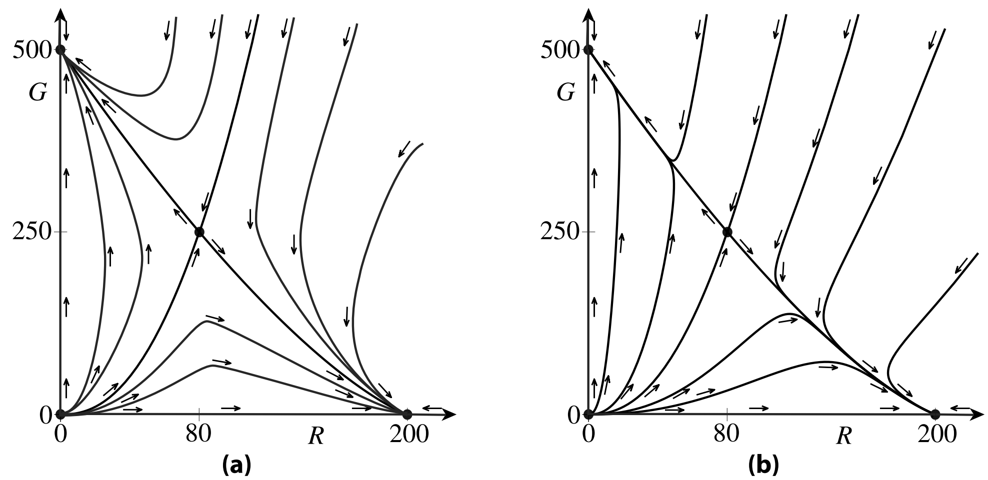


Figure 43.4: A phase portrait for the rabbit/gerbil system of example 43.6 **(a)** "hand drawn" using the derived information and **(b)** "computer drawn".

in the previous paragraph from our rough sketch in figure 43.4a. All the more accurate drawing does for us is to refine our knowledge of the shapes of the trajectories.)

General Analysis of the Competing Species Model

It is worthwhile to redo the analysis just done in the last example, but with the general system for the basic competing species model,

$$\begin{aligned}\frac{dR}{dt} &= (\beta_1 - \gamma_1 R - \alpha_1 G)R \\ \frac{dG}{dt} &= (\beta_2 - \gamma_2 G - \alpha_2 R)G\end{aligned} \quad . \tag{43.18}$$

Remember the β_j's, γ_j's and α_j's are all positive.

Fundamental Features Common to All Competing Species Models

Following the suggestions given in section 43.4, we first compute the Jacobian matrix for our system, obtaining

$$\mathbf{J}(R,G) = \begin{bmatrix} \beta_1 - 2\gamma_1 R - \alpha_1 G & -\alpha_1 R \\ -\alpha_2 G & \beta_2 - 2\gamma_2 G - \alpha_2 R \end{bmatrix} \quad . \tag{43.19}$$

The critical points are then found by solving the algebraic system

$$0 = (\beta_1 - \gamma_1 R - \alpha_1 G)R$$

$$0 = (\beta_2 - \gamma_2 G - \alpha_2 R)G$$

which, because of the factoring of these two equations, is equivalent to finding the solutions to each of the following systems:

$$\begin{aligned} 0 &= R \\ 0 &= G \end{aligned} \quad , \quad \begin{aligned} 0 &= \beta_1 - \gamma_1 R - \alpha_1 G \\ 0 &= G \end{aligned} \quad , \quad \begin{aligned} 0 &= R \\ 0 &= \beta_2 - \gamma_2 G - \alpha_2 R \end{aligned}$$

and

$$\begin{aligned} 0 &= \beta_1 - \gamma_1 R - \alpha_1 G \\ 0 &= \beta_2 - \gamma_2 G - \alpha_2 R \end{aligned} \quad . \tag{43.20}$$

The first three are very easily solved, and, respectively, give us the critical points

$$(0,0) \quad , \quad (R_0, 0) \quad \text{and} \quad (0, G_0)$$

where

$$R_0 = \frac{\beta_1}{\gamma_1} > 0 \quad \text{and} \quad G_0 = \frac{\beta_2}{\gamma_2} > 0 \quad .$$

We'll deal with the possible critical point(s) arising from system (43.20) later, after looking at the behavior of the trajectories around the above critical points.

At critical point $(0,0)$, formula (43.19) for the Jacobian reduces to

$$\mathbf{J}(0,0) = \begin{bmatrix} \beta_1 & 0 \\ 0 & \beta_2 \end{bmatrix} \quad ,$$

which has eigenpairs $(\beta_1, [1,0]^{\mathsf{T}})$ and $(\beta_2, [0,1]^{\mathsf{T}})$. Since $\beta_1 > 0$ and $\beta_2 > 0$, we can immediately conclude that $(0,0)$ is always an unstable node.

At critical point $(R_0, 0)$, formula (43.19) for the Jacobian reduces to

$$\mathbf{J}(R_0, 0) = \begin{bmatrix} -\beta_1 & -\alpha_1 R_0 \\ 0 & \beta_2 - \alpha_2 R_0 \end{bmatrix} \quad .$$

The two eigenvalues of this matrix are the real values $-\beta_1$ and $\beta_2 - \alpha_2 R_0$. Since $-\beta_1 < 0$ this critical point will be

1. a stable node if $\beta_2 - \alpha_2 R_0 < 0$, or

2. a saddle point if $\beta_2 - \alpha_2 R_0 > 0$.[2]

Similarly, at critical point $(0, G_0)$, formula (43.19) for the Jacobian reduces to

$$\mathbf{J}(0, G_0) = \begin{bmatrix} \beta_1 - \alpha_1 G_0 & 0 \\ -\alpha_2 G_0 & -\beta_2 \end{bmatrix} \quad ,$$

which has real eigenvalues $\beta_1 - \alpha_1 G_0$ and $-\beta_2$. Since $-\beta_2 < 0$ this critical point will be

1. a stable node if $\beta_1 - \alpha_1 G_0 < 0$, or

2. a saddle point if $\beta_1 - \alpha_1 G_0 > 0$.

At this point, let us observe that the origin is always an unstable node, and that the positive R–axis and the positive G–axis each contains exactly one critical point, each of which is either a stable node or a saddle point. Let us also note that, because $R_0 = \beta_1/\gamma_1$ and $G_0 = \beta_2/\gamma_2$, the system we are studying (system (43.18)) can be rewritten as

$$\begin{aligned} \frac{dR}{dt} &= (\gamma_1[R_0 - R] - \alpha_1 G)R \\ \frac{dG}{dt} &= (\gamma_2[G_0 - G] - \alpha_2 R)G \end{aligned} \quad . \tag{43.21}$$

Using this system just as we used system (43.16) on page 43–18, you can easily verify the following:

[2] We are ignoring the remote possibility that $\beta_2 - \alpha_2 R_0 = 0$ because the analysis developed in this chapter required that the eigenvalues be nonzero.

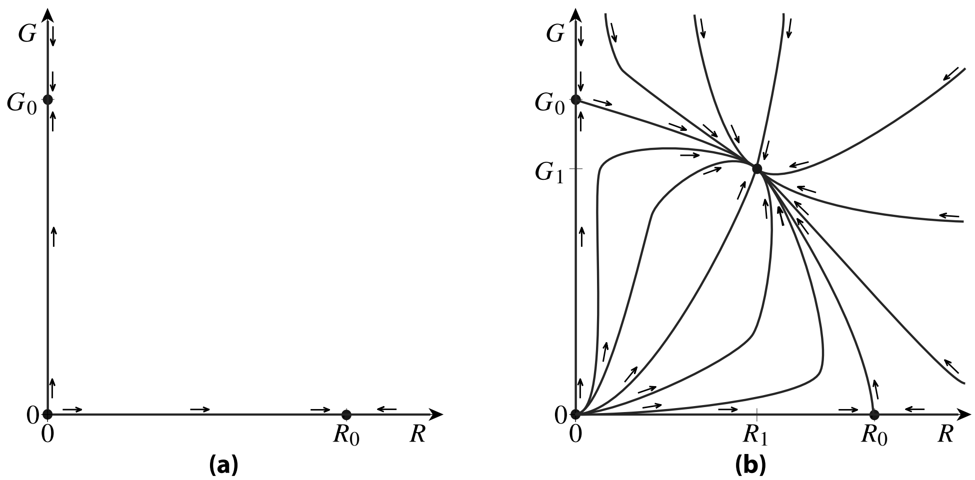


Figure 43.5: Phase portraits for a competing species system: **(a)** The trajectories common to all phase portraits. **(b)** A phase portrait for a system having a "peaceful coexistence" equilibrium.

1. There is one trajectory along the positive R–axis from the origin to the critical point $(R_0, 0)$, and another trajectory towards this critical point along the rest of the positive R–axis.

2. There is one trajectory along the positive G–axis from the origin to the critical point $(0, G_0)$, and another trajectory towards this critical point along the rest of the positive G–axis.

3. The horizontal component of the direction arrow at any point (R, G) with $R \geq R_0$ and $G > 0$ is negative, and, hence, the direction of travel of any trajectory through this point is towards the left.

4. The vertical component of the direction arrow at any point (R, G) with $R > 0$ and $G \geq G_0$ is negative, and, hence, the direction of travel of any trajectory through this point is downwards.

Consequently, no matter what positive values we may have for the β_j's, γ_j's and α_j's, we can at least sketch the partial phase portrait given in figure 43.5a, and, just as in our last example, we are justified in restricting our attention to the region with $R \geq 0$ and $G \geq 0$.

Critical Point(s) Off the Axes (If Any)

Now let's turn our attention to the possible critical points given by algebraic system (43.20) on page 43–22. Since this is a algebraic system of two variables and two linear equations, there are three cases to consider:

1. This linear system is nondegenerate with its one solution (R_1, G_1) in the first quadrant.

2. This linear system has no solutions in the first quadrant.

3. The linear system is degenerate because the two equations in the system are constant multiples of each other.

To simplify things slightly, let us rewrite both that algebraic system and the Jacobian matrix in terms of R_0 and G_0 using the fact that $R_0 = {}^{\beta_1}/_{\gamma_1}$ and $G_0 = {}^{\beta_2}/_{\gamma_2}$. You can easily verify that system (43.20) becomes

$$\begin{aligned} 0 &= \gamma_1[R_0 - R] - \alpha_1 G \\ 0 &= \gamma_2[G_0 - G] - \alpha_2 R \end{aligned} \quad , \tag{43.22}$$

and that the Jacobian matrix (formula 43.19 on page 43–21) becomes

$$\mathbf{J}(R,G) = \begin{bmatrix} \gamma_1[R_0 - 2R] - \alpha_1 G & -\alpha_1 R \\ -\alpha_2 G & \gamma_2[G_0 - 2]G - \alpha_2 R \end{bmatrix} \quad .$$

Now observe that, if (R, G) satisfies system (43.20), then this Jacobian matrix simplifies to

$$\mathbf{J}(R,G) = \begin{bmatrix} -\gamma_1 R & -\alpha_1 R \\ -\alpha_2 G & -\gamma_2 G \end{bmatrix} \quad . \tag{43.23}$$

Now let's look at each of the three cases:

Case 1: Suppose system (43.22) has a single solution (R_1, G_1) in the first quadrant (so $R_1 > 0$ and $G_1 > 0$). Using a little basic linear algebra or by simply solving for (R_1, G_1) you can verify that the nondegeneracy of the system means that

$$\gamma_1\gamma_2 - \alpha_1\alpha_2 \neq 0 \quad .$$

Now, had we specific values for the constants in the system, we would explicitly solve for R_1 and G_1. Here, though, the attempt would only yield cumbersome formulas for R_1 and G_1. Instead, let's make use of the fact that, being a solution to system (43.22), (R_1, G_1) must satisfy

$$0 = \gamma_1[R_0 - R_1] - \alpha_1 G_1 \qquad \text{and} \qquad 0 = \gamma_2[G_0 - G_1] - \alpha_2 R_1 \quad .$$

Rearranging these equations and using the fact that this point is in the first quadrant then gives us

$$R_0 - R_1 = \frac{\alpha_1}{\gamma_1} G_1 > 0 \qquad \text{and} \qquad G_0 - G_1 = \frac{\alpha_2}{\gamma_2} R_2 > 0 \quad ,$$

which, in turn, tells us that

$$0 < R_1 < R_0 \qquad \text{and} \qquad 0 < G_1 < G_0 \quad .$$

So we know roughly where to plot the critical point (R_1, G_1).

Computing the eigenvalues of $\mathbf{J}(R_1, G_1)$ is straightforward but tedious. You do it:

?►Exercise 43.1: *Assume that system (43.22) is nondegenerate, and that R and G are both positive. Verify that the eigenvalues r_+ and r_- for the matrix $\mathbf{J}(R, G)$ in formula (43.23) are both nonzero, and are given by*

$$r_\pm = \frac{-A \pm \sqrt{B}}{2}$$

where

$$A = \gamma_1 R + \gamma_2 G > 0 \qquad \textit{and} \qquad B = (\gamma_1 R - \gamma_2 G)^2 + 4\alpha_1\alpha_2 RG > 0 \quad .$$

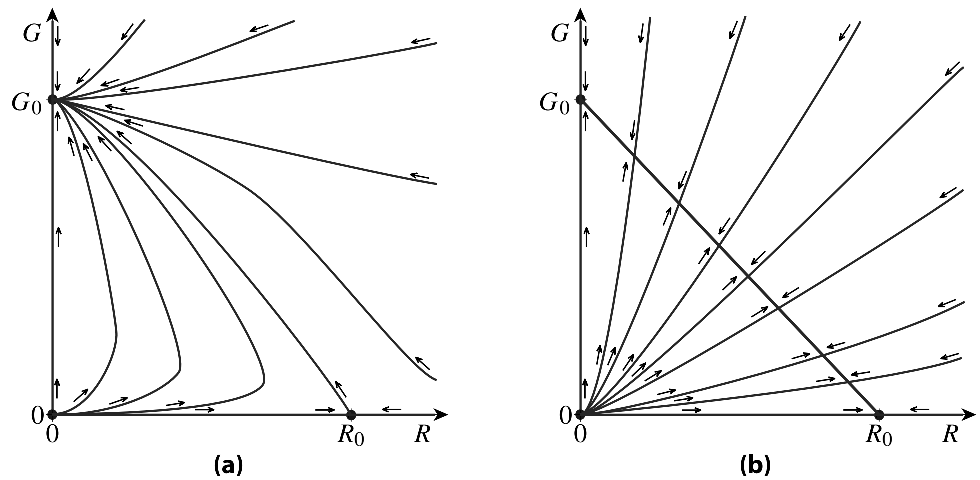


Figure 43.6: Phase portraits for two competing species systems: **(a)** A system having no critical points inside the first quadrant. **(b)** A system having a line of critical points in the first quadrant.

It immediately follows from the above exercise (with $(R, G) = (R_1, G_1)$) that the eigenvalues r_+ and r_- for the $\mathbf{J}(R_1, G_1)$ are real and nonzero, with $r_- < 0$. Consequently, there are two possibilities:

1. If $r_+ > 0$, then critical point (R_1, G_1) is a saddle point, as was illustrated in our earlier example (and graphically illustrated in figure 43.4 on page 43–20). In this case, the eventual outcome — whether we end up with just rabbits or just gerbils — depends on just how many of each we start with.

2. If $r_- < 0$, then critical point (R_1, G_1) is a stable node. When this happens, you should expect a phase portrait similar to that given in figure 43.5b. In this case, the critical points $(R_0, 0)$ and $(0, G_0)$ are saddle points, and this phase portrait tells us that if we start with positive numbers of both rabbits and gerbils, then the populations stabilize at R_1 rabbits and G_1 gerbils as $t \to \infty$. This situation is sometimes referred to as "peaceful coexistence" since neither population overwhelms the other.

Case 2: If system (43.22) has no solution in the first quadrant, then the only critical points of interest are $(0, 0)$, $(R_0, 0)$ and $(0, G_0)$. We know $(0, 0)$ is an unstable node, and that each of the other two can be a saddle point or a stable node. It turns out that, generally, one of those critical points is a stable node and one is a saddle point, as in figure 43.6a. You can get a crude idea of why this is so by attempting to sketch phase portraits with just these three critical points, with both $(R_0, 0)$ and $(0, G_0)$ being saddle points or both being stable nodes.

Observe that, in this case, no matter how many rabbits and gerbils you start with (as long as you start with positive numbers of each), one particular species is forordained to die out. In particular, the model illustrated by figure 43.6a predicts that, in the end, there will only be gerbils (provided, of course, that you started with a few gerbils).

Case 3: Finally, if both equations in system (43.22) are equivalent, then every point on the line

$$0 = \gamma_1 [R_0 - R] - \alpha_1 G$$

is a critical point, and it is easily checked that

$$\gamma_1\gamma_2 - \alpha_1\alpha_2 = 0 \quad ,$$

and that redoing computations done in exercise 43.1 yields

$$r_- = -(\gamma_1 R + \gamma_2 G) \qquad \text{and} \qquad r_+ = 0$$

as the eigenvalues for $\mathbf{J}(R, G)$ at each of these critical points on the above line. Technically, this means that the Jacobian matrix at each of these points is singular and that the analysis developed in the few previous sections does not apply. Still, especially if you've done some of the homework concerning degenerate linear systems (see exercise 39.13 on page 39–29) and you think a little about the possible trajectories, it should not surprise you that the phase portrait for such a system looks something like that sketched in figure 43.6b.

43.6 Application: The Damped Pendulum

In section 36.4 on page 36–14 we derived the system

$$\begin{aligned} \frac{d\theta}{dt} &= \omega \\ \frac{d\omega}{dt} &= -\gamma \sin(\theta) - \kappa\omega \end{aligned} \quad . \tag{43.24}$$

to describe the angular motion of the pendulum in figure 43.7. Here

$\theta(t) =$ the angular position of pendulum at time t measured counterclockwise from the vertical line "below" the pivot point

and

$$\omega(t) = \frac{d\theta}{dt} = \text{the angular velocity of the pendulum at time } t \quad .$$

In addition, γ is a positive constant given by $\gamma = g/L$ where L is the length of the pendulum and g is the acceleration of gravity, and κ is the "drag coefficient", a nonnegative constant describing the effect friction has on the motion of the pendulum. The greater the effect of friction on the system, the larger the value of κ, with $\kappa = 0$ when there is no friction slowing down the pendulum. We will assume $\kappa > 0$ for the analysis here.

Before going any further, do observe that the right side of our system is periodic with period 2π with respect to θ. This means that, on the $\theta\omega$–plane, the pattern of the trajectories in any vertical strip of width 2π will be repeated in the next vertical strip of width 2π.

!►Example 43.7: *To simplify our discussion, let's choose some convenient values for γ and κ, say,*

$$\gamma = 8 \qquad \text{and} \qquad \kappa = 2 \quad .$$

With these values, pendulum system (43.24) becomes

$$\begin{aligned} \theta' &= \omega \\ \omega' &= -8\sin(\theta) - 2\omega \end{aligned} \quad . \tag{43.25}$$

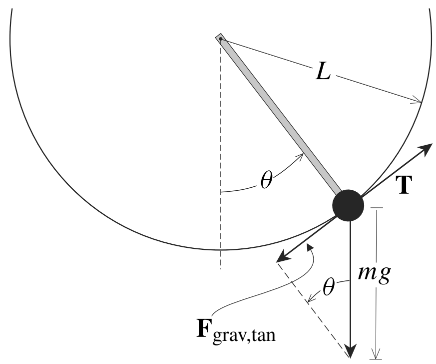


Figure 43.7: The pendulum system with a weight of mass m attached to a massless rod of length L swinging about a pivot point under the influence of gravity.

Computing the Jacobian matrix of this system, we see that

$$\mathbf{J}(\theta,\omega) = \begin{bmatrix} \frac{\partial}{\partial\theta}[\omega] & \frac{\partial}{\partial\omega}[\omega] \\ \frac{\partial}{\partial\theta}[-8\sin(\theta) - 2\omega] & \frac{\partial}{\partial\omega}[-8\sin(\theta) - 2\omega] \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ -8\cos(\theta) & -2 \end{bmatrix} \quad .$$

Setting $\theta' = 0$ and $\omega' = 0$ in system (43.25) yields the algebraic system

$$0 = \omega$$
$$0 = -8\sin(\theta) - 2\omega$$

for the critical points. From this we get that there are infinitely many critical points, and they are given by

$$(\theta,\omega) = (n\pi, 0) \qquad \textit{with} \quad n = 0, \pm 1, \pm 2, \ldots \quad .$$

Thus, the Jacobian matrix at each critical point $(n\pi, 0)$ is

$$\mathbf{J}(n\pi,0) = \begin{bmatrix} 0 & 1 \\ -8\cos(n\pi) & -2 \end{bmatrix} \qquad \textit{for} \quad n = 0, \pm 1, \pm 2, \ldots \quad ,$$

which depends entirely on whether n is even or odd.

If n is even, then $\cos(n\pi) = 1$,

$$\mathbf{A} = \mathbf{J}(n\pi,0) = \begin{bmatrix} 0 & 1 \\ -8 & -2 \end{bmatrix} \quad ,$$

and the corresponding linearized system is

$$\begin{bmatrix} \theta' \\ \omega' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -8 & -2 \end{bmatrix} \begin{bmatrix} \theta - n\pi \\ \omega - 0 \end{bmatrix} \quad .$$

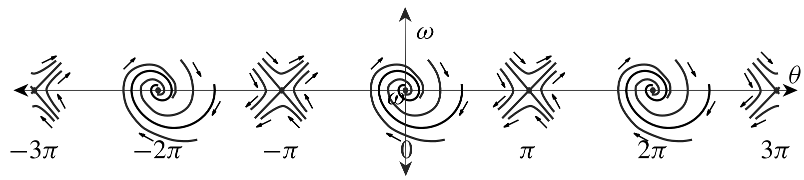


Figure 43.8: Rough sketches of the trajectories of the linearizations of pendulum system (43.25) about the critical points.

Writing out the characteristic equation for $\mathbf{A}$ *, we get*

$$0 = \det\left(\begin{bmatrix} 0 & 1 \\ -8 & -2 \end{bmatrix} - r\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{vmatrix} 0-r & 1 \\ -8 & -2-r \end{vmatrix} = r^2 + 2r + 8 \quad .$$

So the eigenvalues are

$$r_\pm = \frac{-2 \pm \sqrt{(-2)^2 - 4\cdot 8}}{2} = -1 \pm i\sqrt{7} \quad ,$$

a complex conjugate pair with nonzero imaginary parts and negative real parts. This means that these critical points are stable spiral points. The question now is whether they is whether the trajectories spiral in clockwise or counterclockwise. To determine this, let's take the critical point $(n\pi, 0)$ *with* n *being our favorite even integer, namely,* $n = 0$ *. Picking a point on the positive* θ*–axis, say* $(\theta, \omega) = (1, 0)$ *, we see that the direction of the direction arrow there is given by*

$$\begin{bmatrix} \theta' \\ \omega' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -8 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -8 \end{bmatrix} \quad ,$$

which is downward. Hence, when we sketch the trajectories spiraling in to the origin, we find that we must spiral clockwise towards the origin, as illustrated at $\theta = 0$ *in figure 43.8. And because of the aforementioned periodicity, we know that the trajectories at the other critical points with* θ *being an even multiple of* π *.*

Now consider what these spirals are telling us about the angular position $\theta(t)$ *and angular velocity* $\omega(t)$ *of our pendulum as* t *increases. It tells us that if, for some* t_0 *and even integer* n *,*

$$\text{the angular position } \theta(t_0) \text{ is close to } n\pi$$

and

$$\text{the angular speed, } |\omega(t_0)| \text{ is not too big} \quad ,$$

then

$$(\theta(t), \omega(t)) \rightarrow (n\pi, 0) \qquad \text{as} \qquad t \rightarrow \infty \quad .$$

But (with n *even),* $(\theta, \omega) = (n\pi, 0)$ *describes a pendulum hanging straight down and not moving — certainly what most of us would call a 'stable equilibrium' position for the pendulum, and certainly the position we would expect to finally see in a real-world pendulum in which there is inevitably some friction slowing the pendulum.*

Now consider any critical point $(\theta, \omega) = (n\pi, 0)$ *when* n *is an odd integer. In this case,* $\cos(n\pi) = -1$ *,*

$$\mathbf{A} = \mathbf{J}(n\pi, 0) = \begin{bmatrix} 0 & 1 \\ 8 & -2 \end{bmatrix} \quad .$$

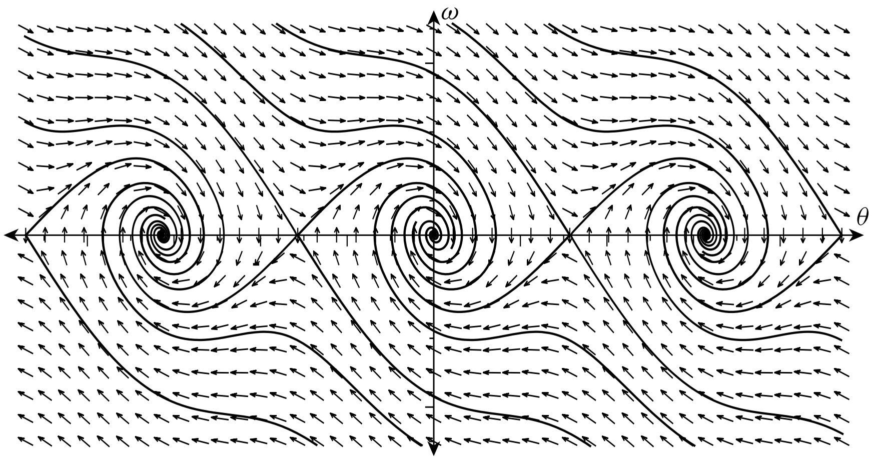


Figure 43.9: A phase portrait (with direction field) for pendulum system (43.25). (Compare this to figure 43.8.)

and

$$0 = \det(\mathbf{A} - r\mathbf{I}) = \begin{vmatrix} 0-r & 1 \\ 8 & -2-r \end{vmatrix} = r^2 + 2r - 8 = (r+4)(r-2) \quad .$$

So the eigenvalues are

$$r_1 = -4 \quad \text{and} \quad r_2 = 2 \quad ,$$

telling us that these critical points are saddle points. It is then an easy matter to show that $[1, -4]^{\mathsf{T}}$ *and* $[1, 2]^{\mathsf{T}}$ *are, respectively, corresponding eigenvectors, giving us the eigenpairs*

$$\left(-4, \begin{bmatrix} 1 \\ -4 \end{bmatrix}\right) \quad \text{and} \quad \left(2 \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) \quad .$$

Applying what we learned in chapter 39.5 regarding the trajectories about saddle points leads to the rough sketches of trajectories in figure 43.8 about the points where $\theta = \pm\pi, \pm 3\pi, \ldots$ *.*

In this case (with n *odd), our critical points are unstable. With* n *being an odd integer,* $(\theta, \omega) = (n\pi, 0)$ *describes a stationary pendulum balanced straight up from its pivot point. Since there are two trajectories leading to each of these critical points, it is theoretically possible to start the pendulum moving in such a manner that it approaches this configuration. But if the initial conditions are not just right, then the motion will be given by a trajectory that approaches and then goes away from that critical point.*

For a more complete picture of the trajectories of our system, we can refine the sketch made in figure 43.8 by analyzing the general directions of the direction arrows in various portions of the $\theta\omega$*–plane, or by just employing a computer-generated direction field. For brevity, we'll turn to the computer. The result is given in figure 43.9.*

It's fairly easy to redo the above using fairly arbitrary choices of γ and κ in pendulum system (43.24). As long as the friction is not too strong (i.e., as long as κ is not too large compared to γ), the resulting phase portrait will be quite similar to what we just obtained. However, if κ is too large compared to γ , then none of the critical points are spiral points. As an exercise, you should figure out just what "too large" means and just what the phase portrait then becomes (see exercise 43.8).

Additional Exercises

43.2. *For each of the following systems:*

i. *Find the Jacobian matrix of the system.*
ii. *Find all the critical points.*
iii. *Compute the Jacobian matrix at each critical point.*
iv. *Write out the corresponding linearized system at each critical point.*
v. *Find the eigenvalues of the matrix of the linearized system at each critical point, and state what can be determined regarding the stability and type (note, saddle point, spiral, etc.) of each critical point from the eigenvalues.*

a. $\begin{aligned} x' &= (x-1)(y-3) \\ y' &= y - x \end{aligned}$

b. $\begin{aligned} x' &= (x-3)(y-1) \\ y' &= y - x \end{aligned}$

c. $\begin{aligned} x' &= x\left(5 - x^2 - y\right) \\ y' &= 4(x^2 - 4) \end{aligned}$

d. $\begin{aligned} x' &= x\left(5 - x^2 + y\right) \\ y' &= 2(x^2 - 4) \end{aligned}$

e. $\begin{aligned} x' &= x\left(5 - x^2 + y\right) \\ y' &= 4 - x^2 \end{aligned}$

f. $\begin{aligned} x' &= (x-2)(y-6) \\ y' &= (x+2)(y-2) \end{aligned}$

g. $\begin{aligned} x' &= x^2 - 4y^2 \\ y' &= (x-2)(y-4) \end{aligned}$

h. $\begin{aligned} x' &= x^2 - 4x - y \\ y' &= y - \frac{5}{2}\left(x^2 - 4x\right) \end{aligned}$

i. $\begin{aligned} x' &= x^2 - 4x - y \\ y' &= y - \frac{3}{2}\left(x^2 - 4x\right) \end{aligned}$

j. $\begin{aligned} x' &= x^2 - 4x - y \\ y' &= 4y - 5\left(x^2 - 4x\right) \end{aligned}$

k. $\begin{aligned} x' &= 2x + \sin(y) \\ y' &= x\left(y^2 + 1\right) \end{aligned}$

43.3. *In this exercise, you will analyze the trajectories of the system*

$$\begin{aligned} x' &= \left(x^2 - 1\right)\left(y^2 + 1\right) \\ y' &= \beta x y \end{aligned}$$

for different values of β .

a. *Assuming* $\beta \neq 0$, *do the following:*

i. *Find the critical points of the above system.*

ii. *Find the Jacobian matrix at each of the critical points.*

iii. *Find the direction arrows at each point on the* Y–*axis (excluding critical points).*

iv. *Find the direction arrows at each point on the* X–*axis (excluding critical points).*

v. *Find the direction arrows at each point* (x, y) *with* $x = \pm 1$ *(excluding critical points).*

b. *For each of the following values of β, determine the stability and type of each of the critical point of the above system, and sketch (by hand) a corresponding phase portrait, noting the subtle (and not-so-subtle) differences between the phase portraits.*

i. $\beta = 2$ **ii.** $\beta = 1$ **iii.** $\beta = 4$ **iv.** $\beta = -4$

43.4. *Each of the following refers to a system from exercise set 43.2, above. For each, sketch (by hand) a rough phase portrait for the system. Use what you've already obtained in the prior exercise, possibly also computing appropriate eigenvectors, direction arrows, etc. (You may want to compare your rough sketch to a computer-generated phase portrait using your favorite software.)*

a. *The system from exercise 43.2 a.*

b. *The system from exercise 43.2 b.*

c. *The system from exercise 43.2 c.*

d. *The system from exercise 43.2 e.*

e. *The system from exercise 43.2 h.*

43.5. *Each of the following systems can be viewed as an example of competing species model from section 43.5 with various values chosen for the parameters in system (43.13). For each, analyze the behavior of (x, y) as $t \to \infty$ after finding and identifying the critical points, and sketching rough phase portraits for the given system.*

a. $$\begin{aligned} R' &= (160 - R - 3G)R \\ G' &= (120 - 2G - R)G \end{aligned}$$

b. $$\begin{aligned} R' &= (120 - 2R - 2G)R \\ G' &= (320 - 8G - 3R)G \end{aligned}$$

c. $$\begin{aligned} R' &= (240 - 3R - 4G)R \\ G' &= (300 - 4G - 3R)G \end{aligned}$$

d. $$\begin{aligned} R' &= (120 - 4R - 4G)R \\ G' &= (160 - 4G - 9R)G \end{aligned}$$

e. $$\begin{aligned} R' &= (120 - 4R - 2G)R \\ G' &= (60 - 2G - R)G \end{aligned}$$

f. $$\begin{aligned} R' &= (50 - R - G)R \\ G' &= (75 - G - 2R)G \end{aligned}$$

g. $$\begin{aligned} R' &= (180 - 2R - 3G)R \\ G' &= (50 - G - R)G \end{aligned}$$

43.6. *Consider the competing species model analyzed in example 43.6.*

a. *Show that the direction arrow at any point on the vertical line segment*

$$\mathcal{v} = \{(R, G) : R = 80\,,\ 0 < G < 250\}$$

points to the right.

b. *Show that the direction arrow at any point on the horizontal line segment*

$$\mathcal{h} = \{(R, G) : 80 < R\,,\ G = 250\}$$

points downward.

c. *Why does the above (along with the analysis done in the example) confirm that the trajectory from $(80, 250)$ initially heading "down and to the right" cannot go to any critical point of the system other than $(200, 0)$.*

43.7. *For each of the following, assume the critical points of some standard "competing species" model,*

$$R' = (\beta_1 - \gamma_1 R - \alpha_1 G)\, R$$
$$G' = (\beta_2 - \gamma_2 G - \alpha_2 R)\, G \qquad ,$$

are as described (with $R(t)$ and $G(t)$ being, respectively, the number of rabbits and gerbils in a large field at time t).

For each:

i. *Sketch a rough phase portrait*

ii. *Describe what happens to $R(t)$ and $G(t)$ as $t \to \infty$, assuming we start with positive values for $R(t)$ and $G(t)$ at $t = 0$.*

a. *The critical points are $(0,0)$, $(50,0)$, $(0,40)$ and $(40,30)$.*
At $(0,0)$, the linearized system has eigen-pairs $\left(2, [1,0]^{\mathsf{T}}\right)$ and $\left(3, [0,1]^{\mathsf{T}}\right)$.
At $(50,0)$, the linearized system has eigen-pairs $\left(-2, [1,0]^{\mathsf{T}}\right)$ and $\left(-4, [0,1]^{\mathsf{T}}\right)$.
At $(0,40)$, the linearized system has eigen-pairs $\left(-3, [1,0]^{\mathsf{T}}\right)$ and $\left(-2, [0,1]^{\mathsf{T}}\right)$.
At $(40,30)$, the linearized system has eigen-pairs $\left(4, [1,-1]^{\mathsf{T}}\right)$ and $\left(-3, [1,1]^{\mathsf{T}}\right)$.

b. *The critical points are $(0,0)$, $(50,0)$, $(0,40)$ and $(40,30)$.*
At $(0,0)$, the linearized system has eigen-pairs $\left(2, [1,0]^{\mathsf{T}}\right)$ and $\left(2, [0,1]^{\mathsf{T}}\right)$.
At $(50,0)$, the linearized system has eigen-pairs $\left(-2, [1,0]^{\mathsf{T}}\right)$ and $\left(4, [0,1]^{\mathsf{T}}\right)$.
At $(0,40)$, the linearized system has eigen-pairs $\left(3, [1,0]^{\mathsf{T}}\right)$ and $\left(-2, [0,1]^{\mathsf{T}}\right)$.
At $(40,30)$, the linearized system has eigen-pairs $\left(-4, [1,-1]^{\mathsf{T}}\right)$ and $\left(-3, [1,1]^{\mathsf{T}}\right)$.

c. *The critical points are $(0,0)$, $(50,0)$ and $(0,40)$.*
At $(0,0)$, the linearized system has eigen-pairs $\left(4, [1,0]^{\mathsf{T}}\right)$ and $\left(2, [0,1]^{\mathsf{T}}\right)$.
At $(50,0)$, the linearized system has eigen-pairs $\left(-2, [1,0]^{\mathsf{T}}\right)$ and $\left(4, [0,1]^{\mathsf{T}}\right)$.
At $(0,40)$, the linearized system has eigen-pairs $\left(-3, [1,0]^{\mathsf{T}}\right)$ and $\left(-2, [0,1]^{\mathsf{T}}\right)$.

43.8. *Consider the damped pendulum system*

$$\frac{d\theta}{dt} = \omega$$
$$\frac{d\omega}{dt} = -\gamma \sin(\theta) - \kappa\omega \qquad .$$

(This is the same as system (43.24) on page 43–26.)

a. *Show that this system will have critical points at $(\theta, \omega) = (n\pi, 0)$ where $n = 0, \pm\pi, \pm 2\pi, \ldots$.*

b. *Let n be an even integer and show that the critical point $(n\pi, 0)$ is*

i. *a spiral point if $\kappa^2 < 4\gamma$.* **ii.** *a stable node if $\kappa^2 > 4\gamma$.*

c. *Sketch a phase portrait for the above system assuming $\kappa = 6$ and $\gamma = 5$.*

Some Answers to Some of the Exercises

WARNING! Most of the following answers were prepared hastily and late at night. They have not been properly proofread! Errors are likely!

2a. $\mathbf{J}(x,y) = \begin{bmatrix} y-3 & x-1 \\ -1 & 1 \end{bmatrix}$; critical points: (1, 1) [saddle point, unstable]; (3, 3) [spiral point, unstable]

2b. $\mathbf{J}(x,y) = \begin{bmatrix} y-1 & x-3 \\ -1 & 1 \end{bmatrix}$; critical points: (1, 1) [saddle point, unstable]; (3, 3) [node, unstable];

2c. $\mathbf{J}(x,y) = \begin{bmatrix} 5-3x^2-y & -x \\ 8x & 0 \end{bmatrix}$; critical points: $(-2,1)$ [spiral point, asymptotically stable]; (2, 1) [spiral point, asymptotically stable]

2d. $\mathbf{J}(x,y) = \begin{bmatrix} 5-3x^2+y & x \\ 4x & 0 \end{bmatrix}$; critical points: $(-2,-1)$ [saddle point, unstable]; $(2,-1)$ [saddle point, unstable]

2e. $\mathbf{J}(x,y) = \begin{bmatrix} 5-3x^2+y & x \\ -2x & 0 \end{bmatrix}$; critical points: $(-2,-1)$ [node, asymptotically stable]; $(2,-1)$ [node, asymptotically stable]

2f. $\mathbf{J}(x,y) = \begin{bmatrix} y-6 & x-2 \\ x+2 & y-2 \end{bmatrix}$; critical points: (2, 2) [saddle point, unstable]; $(-2,6)$ [center or spiral point, stability cannot be determined from the eigenvalues]

2g. $\mathbf{J}(x,y) = \begin{bmatrix} 2x & -8y \\ y-4 & x-2 \end{bmatrix}$; critical points: (2, 1) [saddle point, unstable]; $(2,-1)$ [spiral point, unstable]; (8, 4) [node, unstable]; $(-8,4)$ [node, asymptotically stable]

2h. $\mathbf{J}(x,y) = \begin{bmatrix} 2x-4 & -1 \\ 10-5x & 1 \end{bmatrix}$; critical points: (0, 0) [spiral point, asymptotically stable]; (4, 0) [saddle point, unstable]

2i. $\mathbf{J}(x,y) = \begin{bmatrix} 2x-4 & -1 \\ 6-3x & 1 \end{bmatrix}$; critical points: (0, 0) [node, asymptotically stable]; (4, 0) [saddle point, unstable]

2j. $\mathbf{J}(x,y) = \begin{bmatrix} 2x-4 & -1 \\ 20-10x & 4 \end{bmatrix}$; critical points: (0, 0) [center or spiral point, stability cannot be determined by the eigenvalues]; (4, 0) [saddle point, unstable]

2k. $\mathbf{J}(x,y) = \begin{bmatrix} 2 & \cos(y) \\ y^2+1 & 2xy \end{bmatrix}$; critical points: $(0, n\pi)$ for $n = 0, \pm 2, , \pm 4, \ldots$ [spiral points, unstable]; $(0, k\pi)$ for $k = \pm 1, \pm 3, \pm 5, \ldots$ [saddle points, unstable]

3a i. $(\pm 1, 0)$

3a ii. $\mathbf{J}(1,0) = \begin{bmatrix} 2 & 0 \\ 0 & \beta \end{bmatrix}$, $\mathbf{J}(-1,0) = \begin{bmatrix} 2 & 0 \\ 0 & -\beta \end{bmatrix}$

3a iii. They are horizonatal, pointing to the right.

3a iv. They are horizonatal, pointing to the right if $x^2 > 1$, and to the left if $x^2 < 1$.

3a v. They are vertical, pointing up if $\beta xy > 0$, and down if $\beta xy < 0$.

5a. critical points (0, 0) [unstable node], (160, 0) & (0, 60) [stable nodes], (40, 40) [unstable saddle point] ; as $t \to \infty$, either $R \to 0$ or $G \to 0$.

5b. critical points (0, 0) [unstable node], (60, 0) & (0, 40) [unstable saddle points], (32, 28) [stable node] ; as $t \to \infty$, $(R, G) \to (32, 28)$ (peaceful coexistence).

5c. critical points (0, 0) [unstable node], (80, 0) [unstable saddle point], (0, 75) [stable node] ; as $t \to \infty$, $(R, G) \to (0, 75)$.

5d. critical points (0, 0) [unstable node], (0, 40) & (30, 0) [stable nodes], (8, 22) [unstable saddle point] ; as $t \to \infty$, either $R \to 0$ or $G \to 0$.

5e. critical points (0, 0) [unstable node], (30, 0) & (0, 30) [unstable saddle points], (20, 20) [stable node] ; as $t \to \infty$, $(R, G) \to (20, 20)$ (peaceful coexistence).

5f. critical points (0, 0) [unstable node], (50, 0) & (0, 75) [stable nodes], (25, 25) [unstable saddle point] ; as $t \to \infty$, either $R \to 0$ or $G \to 0$.

5g. critical points (0, 0) [unstable node], (90, 0) [stable node], (0, 50) [unstable saddle point] ; as $t \to \infty$, $(R, T) \to (90, 0)$.