42

Nonhomogeneous Linear Systems

Let us now turn our attention from homogeneous linear systems to nonhomogeneous linear systems. Fortunately, the basic theory and important methods for solving nonhomogeneous systems pretty well parallels the basic theory and methods you already know for solving nonhomogeneous linear differential equations.

42.1 General Theory

So let's consider the problem of solving a nonhomogeneous linear system of differential equations

$$\mathbf{x}' = \mathbf{P}\mathbf{x} + \mathbf{g} \quad ,$$

assuming $\mathbf{P}$ is some $N \times N$ continuous matrix-valued function and $\mathbf{g}$ is some vector-valued function on some interval of interest. Unsurprisingly, we will then refer to the correspondin homogeneous system

$$\mathbf{x}' = \mathbf{P}\mathbf{x}$$

as the *corresponding* or *associated homogeneous system*.

A good start is to make a couple of observations regarding any two particular solutions $\mathbf{x}^p$ and $\mathbf{x}^q$ to the nonhomogeneous system, and any solution $\mathbf{x}^0$ to the corresponding homogeneous system. So we are assuming

$$\frac{d\mathbf{x}^p}{dt} = \mathbf{P}\mathbf{x}^p + \mathbf{g} \quad , \quad \frac{d\mathbf{x}^q}{dt} = \mathbf{P}\mathbf{x}^q + \mathbf{g} \quad \text{and} \quad \frac{d\mathbf{x}^0}{dt} = \mathbf{P}\mathbf{x}^0 \quad .$$

Our first observation is that, by the linearity of differentiation and matrix multiplication,

$$\begin{aligned} \frac{d}{dt}\left[\mathbf{x}^q - \mathbf{x}^p\right] &= \frac{d\mathbf{x}^q}{dt} - \frac{d\mathbf{x}^p}{dt} \\ &= \left(\mathbf{P}\mathbf{x}^q + \mathbf{g}\right) - \left(\mathbf{P}\mathbf{x}^p + \mathbf{g}\right) \\ &= \mathbf{P}\mathbf{x}^q - \mathbf{P}\mathbf{x}^p \\ &= \mathbf{P}\left[\mathbf{x}^q - \mathbf{x}^p\right] \quad , \end{aligned}$$

showing that

$$\mathbf{x}^q(t) - \mathbf{x}^p(t) = \text{a solution to the corresponding homogeneous system} \quad .$$

Let me rephrase this:

If $\mathbf{x}^p$ and $\mathbf{x}^q$ are any two solutions to a given nonhomogeneous linear system of differential equations, then

$$\mathbf{x}^q(t) \;=\; \mathbf{x}^p(t) \;+\; \text{a solution to the corresponding homogeneous system} \quad .$$

On the other hand,

$$\begin{aligned}\frac{d}{dt}\left[\mathbf{x}^p + \mathbf{x}^0\right] &= \frac{d\mathbf{x}^p}{dt} + \frac{d\mathbf{x}^0}{dt} \\ &= \left(\mathbf{P}\mathbf{x}^p + \mathbf{g}\right) + \left(\mathbf{P}\mathbf{x}^0\right) \\ &= \mathbf{P}\mathbf{x}^p + \mathbf{P}\mathbf{x}^0 + \mathbf{g} \\ &= \mathbf{P}\left[\mathbf{x}^p + \mathbf{x}^0\right] + \mathbf{g} \quad .\end{aligned}$$

That is,

If $\mathbf{x}^p$ is a particular solution to a given nonhomogeneous linear system of differential equations, then

$$\mathbf{x}^p(t) \;+\; \text{any solution to the corresponding homogeneous system}$$

is also a solution to the given nonhomogeneous system.

If you check, you'll find that we've just repeated the discussion leading to the theorem on general solutions to nonhomogeneous differential equations, theorem 20.1 on page 474, only now we are dealing with linear systems, and have now derived:

Theorem 42.1 (general solutions to nonhomogeneous systems)
A general solution to a given nonhomogeneous $N \times N$ linear system of differential equations is given by

$$\mathbf{x}(t) \;=\; \mathbf{x}^p(t) \;+\; \mathbf{x}^h(t)$$

where $\mathbf{x}^p$ is any particular solution to the nonhomogeneous equation, and $\mathbf{x}^h$ is a general solution to the corresponding homogeneous system.[1]

This theorem assures us that we can construct a general solution for a nonhomogeneous system of differential equations from any single particular solution $\mathbf{x}^p$, provided we know a general solution $\mathbf{x}^h$ for the corresponding homogeneous system. And, as we will soon see, it is often be a good idea to find that $\mathbf{x}^h$ before seeking a particular solution to the nonhomogeneous system, just as it was a good idea to find a general solution to the corresponding homogeneous differential equation before seeking a particular solution to a given nonhomogeneous differential equation.

Keep in mind that the general solution to the corresponding homogeneous system $\mathbf{x}^h(t)$ contains arbitrary constants, while the particular solution $\mathbf{x}^p(t)$ does not. Recalling the basic theory for homogeneous systems, we see that the formula for $\mathbf{x}$ in the above theorem can also be written as either

$$\mathbf{x}(t) \;=\; \mathbf{x}^p(t) \;+\; c_1\mathbf{x}^1(t) \;+\; c_2\mathbf{x}^2(t) \;+\; \cdots \;+\; c_N\mathbf{x}^N(t) \tag{42.1}$$

[1] Many texts refer to the general solution of the corresponding homogeneous system as "the complementary solution".

where $\{\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^N\}$ is a fundamental set of solutions for $\mathbf{x}' = \mathbf{P}\mathbf{x}$, or as

$$\mathbf{x}(t) = \mathbf{x}^p(t) + [\mathbf{X}(t)]\mathbf{c} \tag{42.2}$$

where $\mathbf{X}$ is a fundamental matrix for $\mathbf{x}' = \mathbf{P}\mathbf{x}$.

While we are at it, I should remind you of the superposition principle for nonhomogeneous equations, theorem 20.4 on page 476. This allowed us to construct solutions to certain nonhomogeneous differential equations as linear combinations of solutions to simpler nonhomogeneous equations. Here is the systems version (which you can easily verify yourself):

Theorem 42.2 (principle of superposition for nonhomogeneous systems)
Let $\mathbf{P}$ be an $N \times N$ matrix-valued function and K a positive integer. Assume $\{\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^K\}$ and $\{\mathbf{g}^1, \mathbf{g}^2, \ldots, \mathbf{g}^K\}$ are two sets of K vector-valued functions related over some interval of interest by

$$\frac{d\mathbf{x}^1}{dt} = \mathbf{P}\mathbf{x}^1 + \mathbf{g}^1 \quad , \quad \frac{d\mathbf{x}^2}{dt} = \mathbf{P}\mathbf{x}^2 + \mathbf{g}^2 \quad , \quad \ldots \quad \text{and} \quad \frac{d\mathbf{x}^K}{dt} = \mathbf{P}\mathbf{x}^K + \mathbf{g}^K \quad .$$

Then, for any set of K constants $\{a_1, a_2, \ldots, a_K\}$, a particular solution to

$$\mathbf{x}' = \mathbf{P}\mathbf{x} + \left[a_1\mathbf{g}^1 + a_2\mathbf{g}^2 + \cdots + a_K\mathbf{g}^K\right]$$

is given by

$$\mathbf{x}^p(t) = a_1\mathbf{x}^1(t) + a_2\mathbf{x}^2(t) + \cdots + a_K\mathbf{x}^K(t) \quad .$$

42.2 Method of Undetermined Coefficients / Educated Guess

If our problem is of the form

$$\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{g}$$

where $\mathbf{A}$ is a constant $N \times N$ matrix and $\mathbf{g}$ is a "relatively simple" vector-valued function involving exponentials, polynomials and sinusoidals, then particular solutions can be found by a fairly straightforward adaptation of the "method of educated guess/undetermined coefficients" developed in chapter chapter 21.[2]

Recall that this method begins with a reasonable "first guess" as to the form for a particular solution.

First Guesses

Let's start with an example.

!►*Example 42.1:* *Consider the nonhomogeneous system*

$$\begin{aligned} x' &= x + 2y + 3t \\ y' &= 2x + y + 2 \end{aligned} \quad .$$

[2] You may want to quickly review chapter 21 before reading the rest of this section.

which, in matrix/vector form, is

$$\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{g}$$

with

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{g}(t) = \begin{bmatrix} 3 \\ 0 \end{bmatrix} t + \begin{bmatrix} 0 \\ 2 \end{bmatrix} \quad .$$

In chapter 21, we saw that, if the nonhomogeneous term in a linear differential equation is a polynomial of degree 1*, then our first guess for the form of a particular solution should also be a polynomial of degree* 1

$$at + b$$

with the coefficients a *and* b *to be determined. But in our problem, a particular solution must be a vector, and so it makes since to replace the unknown constants* a *and* b *with unknown constant vectors, "guessing" that there is a particular solution of the form*

$$\mathbf{x}^p(t) = \mathbf{a}t + \mathbf{b} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} t + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \quad .$$

Plugging this into our system (after rewriting it as $\mathbf{g} = \mathbf{x}' - \mathbf{A}\mathbf{x}$ *to simplify computations), we get*

$$\begin{aligned}
\begin{bmatrix} 3 \\ 0 \end{bmatrix} t + \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \mathbf{g}(t) &= \frac{d\mathbf{x}^p}{dt} - \mathbf{A}\mathbf{x}^p(t) \\
&= \frac{d}{dt}\left(\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} t + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \right) - \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \left(\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} t + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \right) \\
&= \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} - \begin{bmatrix} a_1 + 2a_2 \\ 2a_1 + a_2 \end{bmatrix} t - \begin{bmatrix} b_1 + 2b_2 \\ 2b_1 + b_2 \end{bmatrix} \\
&= \begin{bmatrix} -a_1 - 2a_2 \\ -2a_1 - a_2 \end{bmatrix} t + \begin{bmatrix} a_1 - b_1 - 2b_2 \\ a_2 - 2b_1 - b_2 \end{bmatrix} \quad ,
\end{aligned}$$

clearly requiring that a_1*,* a_2*,* b_1 *and* b_2 *satisfy the two linear algebraic systems*

$$\begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} -a_1 - 2a_2 \\ -2a_1 - a_2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} a_1 - b_1 - 2b_2 \\ a_2 - 2b_1 - b_2 \end{bmatrix} \quad .$$

The first is easily solved, and yields

$$a_1 = 1 \quad \text{and} \quad a_2 = -2 \quad .$$

With this, the second algebraic system becomes

$$\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 - b_1 - 2b_2 \\ -2 - 2b_1 - b_2 \end{bmatrix} \quad \left(\text{i.e.,} \begin{bmatrix} b_1 + 2b_2 \\ 2b_1 + b_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \end{bmatrix} \right)$$

whose solution is easily found to be

$$b_1 = -3 \quad \text{and} \quad b_2 = 2 \quad .$$

So, our particular solution is

$$\mathbf{x}^p(t) = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} t + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} t + \begin{bmatrix} -3 \\ 2 \end{bmatrix} \quad .$$

To find the general solution, we need the general solution $\mathbf{x}^h$ *to* $\mathbf{x}' = \mathbf{A}\mathbf{x}$. *Fortunately, in example 39.2 on page 39–4, we found that general solution to be*

$$\mathbf{x}^h(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-t} \quad .$$

So a general solution to our nonhomogeneous system of differential equations is

$$\mathbf{x}(t) = \mathbf{x}^p(t) + \mathbf{x}^h(t) = \begin{bmatrix} 1 \\ -2 \end{bmatrix} t + \begin{bmatrix} -3 \\ 2 \end{bmatrix} + c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-t} \quad .$$

As illustrated in the above example, the only difference between the first guesses here (for systems), and the first guesses given in chapter 21 (for individual equations) is that the unknown coefficients are constant vectors instead of scalar constants. In particular, if, in the above example,

$$\mathbf{g}(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} \quad ,$$

then our first guess would have been

$$\mathbf{x}^p(t) = \mathbf{a} e^{3t} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} e^{3t} \quad ;$$

if

$$\mathbf{g}(t) = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \sin(3t) \quad ,$$

then our first guess would have been

$$\mathbf{x}^p(t) = \mathbf{a}\cos(3t) + \mathbf{b}\sin(3x) = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \cos(3t) + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \sin(3t) \quad ;$$

and if

$$\mathbf{g}(t) = \begin{bmatrix} 3 \\ 2 \end{bmatrix} t^2 e^{4t} \sin(3t) \quad ,$$

then our first guess would have been

$$\mathbf{x}^p(t) = \left(\mathbf{a}^2 t^2 + \mathbf{a}^1 t + \mathbf{a}^0\right) e^{4t} \cos(3t) + \left(\mathbf{b}^2 t^2 + \mathbf{b}^1 t + \mathbf{b}^0\right) e^{4t} \sin(3x) = \cdots \quad .$$

Second and Subsequent Guesses

In chapter 21, we saw that the first guess would fail if it were also a solution to the corresponding homogeneous problem, but that a second guess consisting of the first guess multiplied by the variable would work (provided no term in that guess is a solution to the corresponding homogeneous problem). As illustrated in the next example, the situation is similar — but not completely analogous — for the systems version.

!►*Example 42.2:* *Consider the nonhomogeneous system*

$$\begin{aligned} x' &= x + 2y + 6e^{3t} \\ y' &= 2x + y + 2e^{3t} \end{aligned} \quad .$$

which, in matrix/vector form, is

$$\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{g}$$

with

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{g}(t) = \begin{bmatrix} 6 \\ 2 \end{bmatrix} e^{3t} \quad .$$

The matrix $\mathbf{A}$ is the same as in examples 39.1 and 39.2. From those examples we know that $\mathbf{A}$ has eigenvalues

$$r = 3 \quad \text{and} \quad r = -1 \quad ,$$

and the corresponding homogeneous system has general solution

$$\mathbf{x}^h(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-t} \quad .$$

Since $\mathbf{g}(t) = \mathbf{g}^0 e^{3t}$ for the constant vector $\mathbf{g}^0 = [6, 2]^{\mathsf{T}}$, the "first guess" is that $\mathbf{x}^p$ would be of the form

$$\mathbf{x}^p(t) = \mathbf{a} e^{3t} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} e^{3t} \quad .$$

There should be some concern, however, because of the similarity between this guess and one of the terms in $\mathbf{x}^h$. After all, for some choices of $\mathbf{a}$, this $\mathbf{x}^p$ is a solution to the corresponding homogeneous problem. In particular, we should be concerned that, because the exponential factor is e^{3t} while 3 is an eigenvalue for $\mathbf{A}$, we may obtain a degenerate system for a_1 and a_2 having no solution.

Plugging the above guess into our system, we get

$$\begin{aligned} \begin{bmatrix} 6 \\ 2 \end{bmatrix} e^{3t} = \mathbf{g}(t) &= \frac{d\mathbf{x}^p}{dt} - \mathbf{A}\mathbf{x}^p(t) \\ &= \frac{d}{dt}\left(\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} e^{3t} \right) - \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \left(\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} e^{3t} \right) \\ &= \cdots \\ &= \begin{bmatrix} 2a_1 - 2a_2 \\ -2a_1 + 2a_2 \end{bmatrix} e^{3t} \quad . \end{aligned}$$

So a_1 and a_2 must satisfy the algebraic system

$$\begin{bmatrix} 6 \\ 2 \end{bmatrix} = \begin{bmatrix} 2a_1 - 2a_2 \\ -2a_1 + 2a_2 \end{bmatrix} \quad ,$$

which is clearly an algebraic system having no possible solution. So our concerns about the above guess were justified.

In chapter 21, we constructed second guesses by multiplying the first guess by the variable. Mimicking that here, let us naively try

$$\mathbf{x}^p(t) = \mathbf{a} t e^{3t} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} t e^{3t} \quad .$$

Plugging this into our system, we now get

$$\begin{aligned}\begin{bmatrix}6\\2\end{bmatrix}e^{3t} = \mathbf{g}(t) &= \frac{d\mathbf{x}^p}{dt} - \mathbf{A}\mathbf{x}^p(t)\\ &= \frac{d}{dt}\left(\begin{bmatrix}a_1\\a_2\end{bmatrix}te^{3t}\right) - \begin{bmatrix}1&2\\2&1\end{bmatrix}\left(\begin{bmatrix}a_1\\a_2\end{bmatrix}te^{3t}\right)\\ &= \cdots\\ &= \begin{bmatrix}2a_1-2a_2\\-2a_1+2a_2\end{bmatrix}te^{3t} + \begin{bmatrix}a_1\\a_2\end{bmatrix}e^{3t} \quad ,\end{aligned}$$

requiring that a_1 *and* a_2 *satisfy both*

$$\begin{bmatrix}0\\0\end{bmatrix} = \begin{bmatrix}2a_1-2a_2\\-2a_1+2a_2\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}6\\2\end{bmatrix} = \begin{bmatrix}a_1\\a_2\end{bmatrix} \quad . \tag{42.3}$$

The good news is that each of these systems has one or more solutions, even though the first system is degenerate. Unfortunately, the solution to the second system does not satisfy the first, as required. So our naive second guess is not sufficient.

If you examine how the systems in (42.3) arose from using $\mathbf{a}te^{3t}$ *as our second guess, you may suspect that we should have included lower-order terms,*

$$\mathbf{x}^p(t) = \mathbf{a}te^{3t} + \mathbf{b}e^{3t} = \begin{bmatrix}a_1\\a_2\end{bmatrix}te^{3t} + \begin{bmatrix}b_1\\b_2\end{bmatrix}e^{3t} \quad .$$

Trying this:

$$\begin{aligned}\begin{bmatrix}6\\2\end{bmatrix}e^{3t} = \mathbf{g}(t) &= \frac{d\mathbf{x}^p}{dt} - \mathbf{A}\mathbf{x}^p(t)\\ &= \frac{d}{dt}\left(\begin{bmatrix}a_1\\a_2\end{bmatrix}te^{3t} + \begin{bmatrix}b_1\\b_2\end{bmatrix}e^{3t}\right) - \begin{bmatrix}1&2\\2&1\end{bmatrix}\left(\begin{bmatrix}a_1\\a_2\end{bmatrix}te^{3t} + \begin{bmatrix}b_1\\b_2\end{bmatrix}e^{3t}\right)\\ &= \cdots\\ &= \begin{bmatrix}2a_1-2a_2\\-2a_1+2a_2\end{bmatrix}te^{3t} + \begin{bmatrix}a_1+2b_1-2b_2\\a_2-2b_1+2b_2\end{bmatrix}e^{3t} \quad ,\end{aligned}$$

which is satisfied if and only if our coefficients satisfy both

$$\begin{bmatrix}0\\0\end{bmatrix} = \begin{bmatrix}2a_1-2a_2\\-2a_1+2a_2\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}6\\2\end{bmatrix} = \begin{bmatrix}a_1+2b_1-2b_2\\a_2-2b_1+2b_2\end{bmatrix} \quad .$$

From the first system, we get $a_1 = a_2$ *. With this, the second system can be rewritten as*

$$\begin{aligned}a_1 + 2b_1 - 2b_2 &= 6\\ a_1 - 2b_1 + 2b_2 &= 2\end{aligned} \quad .$$

Adding these two equations together and dividing by 2 *yields*

$$a_1 = \frac{1}{2}[6+2] = 4 \quad .$$

So

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} \quad ,$$

and the last system further reduces to

$$\begin{aligned} 2b_1 - 2b_2 &= 6 - a_1 = 2 \\ -2b_1 + 2b_2 &= 2 - a_1 = -2 \end{aligned} \quad ,$$

which then reduces to

$$b_1 - b_2 = 1 \quad .$$

We can pick any convenient value for either b_1 *or* b_2 *, obtaining the other from this last simple equation. (This arbitrariness in the values of* b_1 *and* b_2 *reflects the fact that* $\mathbf{b}e^{3t}$ *is a solution to the corresponding homogeneous problem when* $b_1 - b_2 = 0$ *.) Keeping* b_2 *arbitrary, we get*

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 1 + b_2 \\ b_2 \end{bmatrix} \quad .$$

Then choosing, for no particular reason, $b_2 = 0$ *gives us*

$$\mathbf{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad .$$

Finally, we have a second guess

$$\mathbf{x}^p(t) = \mathbf{a}te^{3t} + \mathbf{b}e^{3t}$$

that satisfies our nonhomogeneous system for some choices of $\mathbf{a}$ *and* $\mathbf{b}$ *. In particular, this* $\mathbf{x}^p$ *satisfies the nonhomogeneous system if*

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad .$$

Thus,

$$\mathbf{x}^p(t) = \begin{bmatrix} 4 \\ 4 \end{bmatrix} te^{3t} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{3t}$$

is a particular solution to our nonhomogeneous system, and

$$\mathbf{x}(t) = \mathbf{x}^p(t) + \mathbf{x}^h(t) = \begin{bmatrix} 4 \\ 4 \end{bmatrix} te^{3t} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{3t} + c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-t} \qquad (42.4)$$

is a general solution.

In general, a "first guess" as to the form of a particular solution $\mathbf{x}^p$ to a nonhomogeneous system will fail if it or a term in it could be a solution to the corresponding homogeneous system for some choice of the vector coefficients. In that case, a "second guess" for a particular solution can be constructed by multiplying the first guess by t and adding corresponding lower order terms. And if that fails, a "third guess" is constructed by multiplying the second by t and adding corresponding lower order terms. And so on. Eventually, one of these guesses as to the form of the particular solution will lead to a viable particular solution.

42.3 Reduction of Order/Variation of Parameters

The Basic Variation of Parameters Formula

A relatively simple formula for the solution (over an appropriate interval) to

$$\mathbf{x}' = \mathbf{P}\mathbf{x} + \mathbf{g}$$

can be derived using any fundamental matrix $\mathbf{X}$ for the corresponding homogeneous system $\mathbf{x}' = \mathbf{P}\mathbf{x}$ (so $\mathbf{X}' = \mathbf{P}\mathbf{X}$). The derivation is reminiscent of the reduction of order method described in section 13.3 for solving nonhomogeneous differential equations. We start by expressing the yet unknown solution $\mathbf{x}$ to the nonhomogeneous system as

$$\mathbf{x} = \mathbf{X}\mathbf{u}$$

where $\mathbf{u}$ is a vector-valued function to be determined by plugging this formula for $\mathbf{x}$ into our nonhomogeneous system. Doing so (and using the product rule from section 41.5):

$$\mathbf{x}' = \mathbf{P}\mathbf{x} + \mathbf{g}$$

$$\hookrightarrow \quad (\mathbf{X}\mathbf{u})' = \mathbf{P}[\mathbf{X}\mathbf{u}] + \mathbf{g}$$

$$\hookrightarrow \quad \mathbf{X}'\mathbf{u} + \mathbf{X}\mathbf{u}' = [\mathbf{P}\mathbf{X}]\mathbf{u} + \mathbf{g}$$

$$\hookrightarrow \quad [\mathbf{P}\mathbf{X}]\mathbf{u} + \mathbf{X}\mathbf{u}' = [\mathbf{P}\mathbf{X}]\mathbf{u} + \mathbf{g}$$

$$\hookrightarrow \quad \mathbf{X}\mathbf{u}' = \mathbf{g} \quad .$$

But fundamental matrices are invertible. So we can rewrite the last equation as

$$\mathbf{u}' = \mathbf{X}^{-1}\mathbf{g} \quad . \tag{42.5}$$

Defining the "integral of a matrix" to be just the matrix obtained by integrating each component of the matrix, we now clearly have

$$\mathbf{u}(t) = \int \mathbf{u}'(t)\,dt = \int [\mathbf{X}(t)]^{-1}\mathbf{g}(t)\,dt \quad . \tag{42.6}$$

Combined with the initial formula for $\mathbf{x}$, $\mathbf{x} = \mathbf{X}\mathbf{u}$, and with conditions ensuring the existence of $\mathbf{X}$ and the integrability of $[\mathbf{X}(t)]^{-1}\mathbf{g}(t)$, this yields:

Theorem 42.3 (variation of parameters for systems (indefinite integral version))
Let $\mathbf{P}$ be a continuous $N \times N$ matrix-valued functionon an interval (α, β), and let $\mathbf{X}(t)$ be a fundamental matrix over this interval for the homogeneous system

$$\mathbf{x}' = \mathbf{P}\mathbf{x} \quad .$$

Then, for any continuous vector-valued function $\mathbf{g}$ on (α, β), the solution to the nonhomogeneous problem

$$\mathbf{x}' = \mathbf{P}\mathbf{x} + \mathbf{g}$$

is given by

$$\mathbf{x}(t) \;=\; [\mathbf{X}(t)] \int [\mathbf{X}(t)]^{-1} \mathbf{g}(t)\, dt \quad . \tag{42.7}$$

Formula (42.7) is sometimes called the *variation of parameters formula* for the solution to our nonhomogeneous system. You can either memorize it or be able to rederive it as needed using the "systems" version of reduction of order used above. Keep in mind the arbitrary constants that arise when computing indefinite integrals. That means that, in the computation of the integral in formula (42.6), you should account for an arbitrary constant vector $\mathbf{c}$. If you forget this constant, then the resulting $\mathbf{x}$ is just a particular solution. If you include it, the resulting $\mathbf{x}$ is a general solution.

!►Example 42.3: *Let g_1 and g_2 be any pair of functions continuous on $(-\infty, \infty)$, and consider solving the system*

$$\begin{aligned} x' &= x + 2y + g_1 \\ y' &= 2x + y + g_2 \end{aligned} \quad .$$

In matrix/vector form, this is

$$\mathbf{x}' \;=\; \mathbf{A}\mathbf{x} + \mathbf{g}$$

where $\mathbf{A}$ is, again, the matrix from examples 39.1 and 39.2 , and $\mathbf{g} = [g_1, g_2]^{\mathsf{T}}$.

The corresponding homogeneous system is $\mathbf{x}' = \mathbf{A}\mathbf{x}$, which we considered in example 39.2. There we found that

$$\left\{ \begin{bmatrix} e^{3t} \\ e^{3t} \end{bmatrix} , \begin{bmatrix} -e^{-t} \\ e^{-t} \end{bmatrix} \right\}$$

is a fundamental set of solutions to the homogeneous system. Hence,

$$\mathbf{X}(t) \;=\; \begin{bmatrix} e^{3t} & -e^{-t} \\ e^{3t} & e^{-t} \end{bmatrix}$$

is a fundamental matrix for the homogeneous system. It's inverse is easily found,

$$[\mathbf{X}(t)]^{-1} \;=\; \frac{1}{2} \begin{bmatrix} e^{-3t} & e^{-3t} \\ -e^{t} & e^{t} \end{bmatrix} \quad .$$

From the derivation of formula (42.7), we know the solution to our nonhomogeneous system is

$$\mathbf{x}(t) \;=\; [\mathbf{X}(t)] \int [\mathbf{X}(t)]^{-1} \mathbf{g}(t)\, dt \;=\; \frac{1}{2} \begin{bmatrix} e^{3t} & -e^{-t} \\ e^{3t} & e^{-t} \end{bmatrix} \int \begin{bmatrix} e^{-3t} & e^{-3t} \\ -e^{t} & e^{t} \end{bmatrix} \begin{bmatrix} g_1(t) \\ g_2(t) \end{bmatrix} dt \quad .$$

In particular, if

$$\mathbf{g}(t) \;=\; \begin{bmatrix} 6 \\ 2 \end{bmatrix} e^{3t} \quad ,$$

as in example 42.2, then

$$\begin{aligned} \mathbf{x}(t) &= \frac{1}{2} \begin{bmatrix} e^{3t} & -e^{-t} \\ e^{3t} & e^{-t} \end{bmatrix} \int \begin{bmatrix} e^{-3t} & e^{-3t} \\ -e^{t} & e^{t} \end{bmatrix} \begin{bmatrix} 6e^{3t} \\ 2e^{3t} \end{bmatrix} dt \\ &= \frac{1}{2} \begin{bmatrix} e^{3t} & -e^{-t} \\ e^{3t} & e^{-t} \end{bmatrix} \int \begin{bmatrix} 8 \\ -4e^{4t} \end{bmatrix} dt \end{aligned}$$

$$= \frac{1}{2}\begin{bmatrix} e^{3t} & -e^{-t} \\ e^{3t} & e^{-t} \end{bmatrix}\begin{bmatrix} 8t + c_1 \\ e^{4t} + c_2 \end{bmatrix}$$

$$= \cdots$$

$$= \begin{bmatrix} 4 \\ 4 \end{bmatrix} te^{3t} + \frac{1}{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{3t} + c_3\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} + c_4\begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-t} \quad .$$

Comparing this to that obtained in example 42.2, we see that they are the same, with the above c_3 *and* c_4 *related to the* c_1 *and* c_2 *in formula (42.4) on page 42–8 by*

$$c_3 = c_1 + \frac{1}{2} \qquad \text{and} \qquad c_4 = c + 2 \quad .$$

The Definite Integral Version

We should note that, instead of using an indefinite integral with equation (42.5), we could have used the definite integral,

$$\mathbf{u}(t) - \mathbf{u}(t_0) = \int_{s=t_0}^{t} \mathbf{u}'(s)\,ds = \int_{s=t_0}^{t} [\mathbf{X}(s)]^{-1}\mathbf{g}(s)\,ds \quad .$$

Then

$$\mathbf{u}(t) = \mathbf{u}(t_0) + \int_{s=t_0}^{t} [\mathbf{X}(s)]^{-1}\mathbf{g}(s)\,ds \quad ,$$

$$\mathbf{x}(t) = [\mathbf{X}(t)]\mathbf{u}(t) = [\mathbf{X}(t)]\mathbf{u}(t_0) + [\mathbf{X}(t)]\int_{s=t_0}^{t} [\mathbf{X}(s)]^{-1}\mathbf{g}(s)\,ds$$

and

$$\mathbf{x}(t_0) = [\mathbf{X}(t_0)]\mathbf{u}(t_0) + [\mathbf{X}(t_0)]\int_{s=t_0}^{t_0} [\mathbf{X}(s)]^{-1}\mathbf{g}(s)\,ds = [\mathbf{X}(t_0)]\mathbf{u}(t_0) + \mathbf{0} \quad .$$

Solving this last equation for $\mathbf{u}(t_0)$ and plugging it back into the previous equation gives the following version of the variation of parameters formula for systems:

$$\mathbf{x}(t) = [\mathbf{X}(t)][\mathbf{X}(t_0)]^{-1}\mathbf{x}(t_0) + [\mathbf{X}(t)]\int_{s=t_0}^{t} [\mathbf{X}(s)]^{-1}\mathbf{g}(s)\,ds \quad . \tag{42.8}$$

This formula for $\mathbf{x}$ may be preferable to formula (42.7) when dealing with initial-value problems. What's more, the above integral makes sense even if the components of $\mathbf{g}$ are merely piecewise continuous on our interval of interest, suggesting that we can relax our requirement that the components of $\mathbf{g}$ be continuous. And, indeed, we can.

Theorem 42.4 (variation of parameters for systems (definite integral version))
Let $\mathbf{P}$ *be a continuous* $N \times N$ *matrix-valued function over an interval* (α, β) *, and let* $\mathbf{X}(t)$ *be a fundamental matrix over this interval for the homogeneous system*

$$\mathbf{x}' = \mathbf{P}\mathbf{x} \quad .$$

Then, for any vector $\mathbf{g}$ *of functions piecewise continuous on* (α, β) *, any* t_0 *in* (α, β) *and any constant column vector* $\mathbf{x}^0$ *, the solution to the nonhomogeneous initial-value problem*

$$\mathbf{x}' = \mathbf{P}\mathbf{x} + \mathbf{g} \qquad \text{with} \quad \mathbf{x}(t_0) = \mathbf{x}^0$$

is given by

$$\mathbf{x}(t) = [\mathbf{X}(t)][\mathbf{X}(t_0)]^{-1}\mathbf{x}^0 + [\mathbf{X}(t)]\int_{s=t_0}^{t}[\mathbf{X}(s)]^{-1}\mathbf{g}(s)\,ds \quad . \qquad (42.9)$$

That formula (42.9) is a solution to the given initial-value problem can be verified by plugging it into the initial-value problem and using properties of fundamental matrices and basic facts from calculus. To see that this formula is the only solution, go back over the derivation of this formula, and observe that it would still have been obtained assuming $\mathbf{x} = \mathbf{X}\mathbf{u}$ where $\mathbf{u}$ is given by $\mathbf{X}^{-1}$ multiplied on the right by any given solution to the initial-value problem. Hence, any given solution must be given by this one formula. The details of all this will be left to the interested reader.

Enough theory. Let's use formula (42.9) to solve a simple problem involving a step function,

$$\text{step}_\gamma(t) = \begin{cases} 0 & \text{if} \quad t < \gamma \\ 1 & \text{if} \quad \gamma < t \end{cases} \quad .$$

!►*Example 42.4:* *Consider solving the system*

$$\begin{aligned} x' &= x + 2y + 3\,\text{step}_1(t) \\ y' &= 2x + y \end{aligned}$$

with initial conditions

$$x(0) = 0 \qquad \text{and} \qquad y(0) = 2 \quad .$$

In matrix/vector form, this is

$$\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{g} \qquad \text{with} \quad \mathbf{x}(0) = \mathbf{x}^0$$

where $\mathbf{A}$ *is the same matrix used in the last example,*

$$\mathbf{g}(t) = \begin{bmatrix} 3\,\text{step}_2(t) \\ 0 \end{bmatrix} \qquad \text{and} \qquad \mathbf{x}^0 = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \quad .$$

From the last example, we already know both a fundamental matrix $\mathbf{X}$ *for the corresponding homogeneous system, and the inverse of this matrix,*

$$\mathbf{X}(t) = \begin{bmatrix} e^{3t} & -e^{-t} \\ e^{3t} & e^{-t} \end{bmatrix} \qquad \text{and} \qquad [\mathbf{X}(t)]^{-1} = \frac{1}{2}\begin{bmatrix} e^{-3t} & -e^{-3t} \\ -e^{t} & e^{t} \end{bmatrix} \quad .$$

From this, it follows that

$$[\mathbf{X}(0)]^{-1} = \frac{1}{2}\begin{bmatrix} e^{-3\cdot 0} & -e^{-3\cdot 0} \\ -e^{0} & e^{0} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad .$$

So,

$$[\mathbf{X}(t)][\mathbf{X}(0)]^{-1}\mathbf{x}^0 = \begin{bmatrix} e^{3t} & -e^{-t} \\ e^{3t} & e^{-t} \end{bmatrix}\left(\frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix}\right) = \cdots = \begin{bmatrix} -e^{3t} - e^{-t} \\ -e^{3t} + e^{-t} \end{bmatrix}$$

and

$$\int_{s=0}^{t} [\mathbf{X}(s)]^{-1}\mathbf{g}(s)\,ds \;=\; \int_{s=0}^{t} \begin{bmatrix} e^{-3s} & -e^{-3s} \\ -e^{s} & e^{s} \end{bmatrix} \begin{bmatrix} 3\,\text{step}_2(s) \\ 0 \end{bmatrix} ds$$

$$= \;\cdots\; = \; 3\int_{s=0}^{t} \begin{bmatrix} e^{-3s} \\ -e^{s} \end{bmatrix} \text{step}_2(s)\,ds \quad .$$

If $t < 2$*, then*

$$\int_{s=0}^{t} [\mathbf{X}(s)]^{-1}\mathbf{g}(s)\,ds \;=\; 3\int_{s=0}^{t} \begin{bmatrix} e^{-3s} \\ -e^{s} \end{bmatrix} \underbrace{\text{step}_2(s)}_{0}\,ds \;=\; \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad .$$

Thus,

$$[\mathbf{X}(t)]\int_{s=0}^{t} [\mathbf{X}(s)]^{-1}\mathbf{g}(s)\,ds \;=\; [\mathbf{X}(t)]\mathbf{0} \;=\; \mathbf{0} \qquad \text{if} \quad t < 2 \quad ,$$

and applying the variation of parameters formula for the solution $\mathbf{x}$ *to our initial-value problem yields*

$$\mathbf{x}(t) \;=\; [\mathbf{X}(t)][\mathbf{X}(0)]^{-1}\mathbf{x}(0) \;+\; [\mathbf{X}(t)]\int_{s=0}^{t} [\mathbf{X}(s)]^{-1}\mathbf{g}(s)\,ds$$

$$= \begin{bmatrix} -e^{3t} - e^{-t} \\ -e^{3t} + e^{-t} \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} e^{3t} \;+\; \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-t} \qquad \text{if} \quad t < 2 \quad .$$

On the other hand, if $2 < t$*, then*

$$\int_{s=0}^{t} [\mathbf{X}(s)]^{-1}\mathbf{g}(s)\,ds \;=\; 3\int_{s=0}^{2} \begin{bmatrix} e^{-3s} \\ -e^{s} \end{bmatrix} \underbrace{\text{step}_2(s)}_{0}\,ds \;+\; 3\int_{s=2}^{t} \begin{bmatrix} e^{-3s} \\ -e^{s} \end{bmatrix} \underbrace{\text{step}_2(s)}_{1}\,ds$$

$$= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \;+\; \begin{bmatrix} -e^{-3t} + e^{-3\cdot 2} \\ -3e^{t} + 3e^{2} \end{bmatrix} \quad ,$$

and

$$[\mathbf{X}(t)]\int_{s=0}^{t} [\mathbf{X}(s)]^{-1}\mathbf{g}(s)\,ds \;=\; \begin{bmatrix} e^{3t} & -e^{-t} \\ e^{3t} & e^{-t} \end{bmatrix} \begin{bmatrix} -e^{-3t} + e^{-3\cdot 2} \\ -3e^{t} + 3e^{2} \end{bmatrix}$$

$$= \begin{bmatrix} -1 + e^{3(t-2)} + 3 - 3e^{2-t} \\ -1 + e^{3(t-2)} - 3 + 3e^{2-t} \end{bmatrix}$$

$$= \;\cdots$$

$$= \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3(t-2)} \;+\; \begin{bmatrix} -3 \\ 3 \end{bmatrix} e^{2-t} \;+\; \begin{bmatrix} 2 \\ -4 \end{bmatrix} \quad .$$

Thus, when $t > 2$*,*

$$\mathbf{x}(t) \;=\; [\mathbf{X}(t)][\mathbf{X}(0)]^{-1}\mathbf{x}^{0} \;+\; [\mathbf{X}(t)]\int_{s=0}^{t} [\mathbf{X}(s)]^{-1}\mathbf{g}(s)\,ds$$

$$= \begin{bmatrix} -1 \\ -1 \end{bmatrix} e^{3t} \;+\; \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-t} \;+\; \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3(t-2)} \;+\; 3\begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{2-t} \;+\; \begin{bmatrix} 2 \\ -4 \end{bmatrix} \quad .$$

So,

$$\mathbf{x}(t) = \begin{cases} \begin{bmatrix} -1 \\ -1 \end{bmatrix} e^{3t} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-t} & \text{if } t < 2 \\ \begin{bmatrix} -1 \\ -1 \end{bmatrix} e^{3t} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-t} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3(t-2)} + 3 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{2-t} + \begin{bmatrix} 2 \\ -4 \end{bmatrix} & \text{if } 2 < t \end{cases} .$$

42.4 Laplace Transforms

It should be mentioned that a constant matrix system of differential equations can be reduced to a linear system of algebraic equations by taking the Laplace transform of each equation. After solving the resulting algebraic system, the solution to the original system of differential equations can then be found by taking the inverse Laplace transforms of the solutions to the algebraic system.

One example should adequately illustrate the positive and negative points of this approach.

!►*Example 42.5:* *Again, consider solving the system*

$$\begin{aligned} x' &= x + 2y + 3\,\text{step}_1(t) \\ y' &= 2x + y \end{aligned}$$

with initial conditions

$$x(0) = 0 \quad \text{and} \quad y(0) = 2 \quad .$$

Taking the Laplace transform of the first equation:

$$\mathcal{L}\big[x'\big]\big|_s = \mathcal{L}[x]|_s + 2\mathcal{L}[y]|_s + 3\mathcal{L}\big[\text{step}_1(t)\big]\big|_s$$

$$\hookrightarrow \quad sX(s) - x(0) = X(s) + 2Y(s) + \frac{3}{s}e^{-2s}$$

$$\hookrightarrow \quad sX(s) - 0 = X(s) + 2Y(s) + \frac{3}{s}e^{-2s}$$

$$\hookrightarrow \quad [s-1]X(s) - 2Y(s) = \frac{3}{s}e^{-2s}$$

Doing the same with the second equation:

$$\mathcal{L}\big[y'\big]\big|_s = 2\mathcal{L}[x]|_s + \mathcal{L}[y]|_s$$

$$\hookrightarrow \quad sY(s) - y(0) = 2X(s) + Y(s)$$

$$\hookrightarrow \quad sY(s) - 2 = 2X(s) + Y(s)$$

$$\hookrightarrow \quad -2X(s) + [s-1]Y(s) = 2$$

So the transforms $X = \mathcal{L}[x]$ *and* $Y = \mathcal{L}[y]$ *must satisfy the algebraic system*

$$\begin{aligned} [s-1]X(s) - 2Y(s) &= \frac{3}{s}e^{-2s} \\ -2X(s) + [s-1]Y(s) &= 2 \end{aligned} \quad . \tag{42.10}$$

It is relatively easy to solve the above system. To find $X(s)$ *we can first add* $s-1$ *times the first equation to* 2 *times the second to obtain*

$$[s-1]^2 X(s) - 4X(s) = \frac{3(s-1)}{s}e^{-2s} + 4 \quad .$$

After rewriting this as

$$\left[(s-1)^2 - 4\right] X(s) = \frac{3(s-1)}{s}e^{-2s} + 4 \quad ,$$

we see that

$$X(s) = \frac{3(s-1)}{s\left[(s-1)^2-4\right]}e^{-2s} + \frac{4}{(s-1)^2-4} \quad . \tag{42.11}$$

Similarly, to find $Y(s)$ *, we can add* 2 *times the first equation in system (42.10) to* $(s-1)$ *times the second equation, obtaining*

$$-4Y(s) + [s-1]^2 Y(s) = \frac{6}{s}e^{-2s} + 2(s-1) \quad .$$

Solving this for $Y(s)$ *then yields*

$$Y(s) = \frac{6}{s\left[(s-1)^2-4\right]}e^{-2s} + \frac{2(s-1)}{(s-1)^2-4} \quad . \tag{42.12}$$

Finding the formulas for X *and* Y *was easy. Now we need to compute the formulas for* $x = \mathcal{L}^{-1}[X]$ *and* $y = \mathcal{L}^{-1}[Y]$ *from formulas (42.11) and (42.12) using the theory, techniques and tricks for finding inverse Laplace transforms developed in chapters 24 through 28 of this text. That is less easy, and the details will be left to the reader "as a review of Laplace transform" (and to save space here). Suffice it to say that, after using the necessary translation identities, partial fractions or convolutions, the end result will be the same as obtained in example 42.4.*

42.5 Using Similarity Transforms

In section 41.6, we discussed converting a homogeneous constant $N \times N$ matrix system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ to an equivalent homogeneous constant matrix system $\mathbf{y}' = \mathbf{B}\mathbf{y}$ by means of a similarity transform in which $\mathbf{A}$ and $\mathbf{B}$ are related by

$$\mathbf{B} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T} \qquad \text{and} \qquad \mathbf{A} = \mathbf{T}\mathbf{B}\mathbf{T}^{-1}$$

for some invertible $N \times N$ matrix $\mathbf{T}$. The solutions $\mathbf{x}$ and $\mathbf{y}$ are related by

$$\mathbf{y} = \mathbf{T}^{-1}\mathbf{x} \qquad \text{and} \qquad \mathbf{x} = \mathbf{T}\mathbf{y} \quad .$$

This fact was derived in equation sets (41.9) on page 41–16. You can easily redo those computations assuming $\mathbf{x}$ and $\mathbf{y}$ satisfy nonhomogeneous systems, and derive:

Theorem 42.5
Let $\mathbf{A}$ and $\mathbf{B}$ be two constant $N \times N$ matrices related by a similarity transform

$$\mathbf{B} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T} \qquad \text{and} \qquad \mathbf{A} = \mathbf{T}\mathbf{B}\mathbf{T}^{-1} \quad ,$$

and let $\mathbf{x}$ and $\mathbf{y}$ be two vector-valued functions on $(-\infty, \infty)$ related by

$$\mathbf{y} = \mathbf{T}^{-1}\mathbf{x} \qquad \text{and} \qquad \mathbf{x} = \mathbf{T}\mathbf{y} \quad .$$

Then, for any vector-valued function $\mathbf{g}$ on $(-\infty, \infty)$, $\mathbf{x}$ is a solution to

$$\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{g}$$

if and only if $\mathbf{y}$ is a solution to

$$\mathbf{y}' = \mathbf{B}\mathbf{y} + \mathbf{T}^{-1}\mathbf{g} \quad .$$

?►*Exercise 42.1:* *Derive the claim made in the last theorem.*

Of course, for the above to be of value in solving $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{g}$, we should choose the matrix $\mathbf{T}$ so that the corresponding equivalent system $\mathbf{y}' = \mathbf{B}\mathbf{y} + \mathbf{T}^{-1}\mathbf{g}$ is as easily solved as possible. In particular, we would like to choose $\mathbf{T}$ so that $\mathbf{B}$ is as described in theorem 41.3 on page 41–15. And remember, when $\mathbf{A}$ has a complete set of eigenvectors $\{\mathbf{u}^1, \mathbf{u}^2, \cdots, \mathbf{u}^N\}$, this $\mathbf{T}$ can be given by the matrix whose k^{th} column is $\mathbf{u}^k$.

!►*Example 42.6:* *Let us consider, one more time, the nonhomogeneous system considered in examples 42.2 and 42.3; namely,*

$$\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{g}$$

with

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{g}(t) = \begin{bmatrix} 6 \\ 2 \end{bmatrix} e^{3t} \quad .$$

From examples 39.1 and 39.2, we know $\mathbf{A}$ has a complete set of eigenvectors $\{\mathbf{u}^1, \mathbf{u}^2\}$ with

$$\mathbf{u}^1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{u}^2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad .$$

(We also know $\mathbf{A}$ has eigenpairs $(3, \mathbf{u}^1)$ and $(-1, \mathbf{u}^2)$, and that $\{\mathbf{u}^1 e^{3t}, \mathbf{u}^2 e^{-t}\}$ is a fundamental set of solutions for the corresponding homogeneous problem $\mathbf{x}' = \mathbf{A}\mathbf{x}$.)

Computing the matrix $\mathbf{T}$ whose k^{th} column is given by $\mathbf{u}^k$ and its inverse gives us

$$\mathbf{T} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{T}^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \quad .$$

Hence,

$$\mathbf{B} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \cdots = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} \quad ,$$

$$\mathbf{T}^{-1}\mathbf{g} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 6 \\ 2 \end{bmatrix} e^{3t} = \begin{bmatrix} 4 \\ -2 \end{bmatrix} e^{3t} \quad .$$

and, letting $\mathbf{y} = [y_1, y_2]^{\mathsf{T}}$,

$$\mathbf{By} + \mathbf{T}^{-1}\mathbf{g} = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} -2 \\ 4 \end{bmatrix} e^{3t} = \begin{bmatrix} 3y_1 + 4e^{3t} \\ -y_2 - 2e^{3t} \end{bmatrix} \quad .$$

Consequently, the system $\mathbf{y}' = \mathbf{By} + \mathbf{T}^{-1}\mathbf{g}$ *is simply*

$$\begin{bmatrix} y_1{}' \\ y_2{}' \end{bmatrix} = \begin{bmatrix} 3y_1 + 4e^{3t} \\ -y_2 - 2e^{3t} \end{bmatrix} \quad .$$

This is a completely uncoupled system. For convenience, let's rewrite it as

$$\begin{aligned} \frac{dy_1}{dt} - 3y_1 &= 4e^{3t} \\ \frac{dy_2}{dt} + y_2 &= -2e^{3t} \end{aligned} \quad , \tag{42.13}$$

noting that each equation in this system is a simple first-order linear differential equation that we can solve using the methods from chapter 5 (using t *as the variable, instead of* x *).*

For the first equation in system (42.13), we start by finding the integrating factor

$$\mu(t) = e^{\int(-3)\,dt} = e^{-3t} \quad .$$

Then we multiply the differential equation by this integrating factor and proceed as described in section 5.2,

$$e^{-3t}\left[\frac{dy_1}{dt} - 3y_1\right] = e^{-3t}\left[4e^{3t}\right]$$

$$\hookrightarrow \qquad e^{-3t}\frac{dy_1}{dt} - 3e^{-3t}y_1 = 4$$

$$\hookrightarrow \qquad \frac{d}{dt}\left[e^{-3t}y_1\right] = 4$$

$$\hookrightarrow \qquad \int \frac{d}{dt}\left[e^{-3t}y_1\right] dt = \int 4\,dt$$

$$\hookrightarrow \qquad e^{-3t}y_1 = 4t + c_1 \quad .$$

Multiplying both sides of the last equation by e^{3t} *then gives us our formula for* y_1,

$$y_1(t) = 4te^{-3t} + c_1e^{-3t} \quad .$$

For the second equation in system (42.13), the integrating factor is

$$\mu(t) = e^{\int 1\,dt} = e^t \quad .$$

Multiplying the differential equation by this integrating factor and proceeding as before, we get

$$e^t\left[\frac{dy_2}{dt} + y_2\right] = e^t\left[-2e^{3t}\right]$$

$$\hookrightarrow \qquad \frac{d}{dt}\left[e^t y_1\right] = -2e^{4t}$$

$$\hookrightarrow \qquad e^t y_2 = -\frac{1}{2}e^{4t} + c_2 \quad .$$

Thus,

$$y_2(t) = e^{-t}\left[-\frac{1}{2}e^{4t} + c_2\right] = -\frac{1}{2}e^{3t} + c_2 e^{-t} \quad ,$$

and our solution to system (42.13) is

$$\mathbf{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 4te^{-3t} + c_1 e^{-3t} \\ -\frac{1}{2}e^{3t} + c_2 e^{-t} \end{bmatrix} \quad .$$

Finally, we need to compute the formula for $\mathbf{x}$ *from our formula for* $\mathbf{y}$ *:*

$$\begin{aligned} \mathbf{x}(t) = \mathbf{T}\mathbf{y}(t) &= \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 4te^{-3t} + c_1 e^{-3t} \\ -\frac{1}{2}e^{3t} + c_2 e^{-t} \end{bmatrix} \\ &= \cdots \\ &= \begin{bmatrix} 4 \\ 4 \end{bmatrix} te^{-3t} + \frac{1}{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{3t} + c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-3t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-t} \quad . \end{aligned}$$

which, unsurprisingly, is the same as obtained in both example 42.2 and example 42.3.

Whenever $\mathbf{A}$ has a complete set of eigenvectors, then choosing $\mathbf{T}$ as just described will lead to a completely uncoupled system of first-order linear differential equations. If some of the eigenvalues happen to be complex, then the corresponding differential equations will have complex terms, leading to solutions involving complex exponentials. This means that, in the end, you will have to do a little more work to convert your answers to answers involving just real-valued functions.

When $\mathbf{A}$ does not have a complete set of eigenvectors, then you can construct $\mathbf{T}$ from the $\mathbf{w}^{k,j}$'s described in theorems 40.8 and 40.9 from section 40.5. The resulting matrix $\mathbf{B}$ will be as described in theorem 41.3 on page 41–15, and the system $\mathbf{y}' = \mathbf{B}\mathbf{y} + \mathbf{T}^{-1}\mathbf{g}$ will be weakly coupled. Finding the solution to this system will doubtlessly require a bit more labor than in the above example, but will still be relatively straightforward.

Whether or not $\mathbf{A}$ has a complete set of eigenvectors, we still have the question of whether "using similarity transforms" is any better than using, say, either the method of educated guess or the variation of parameters method. Admittedly, it is nice to know that every nonhomogeneous linear $N \times N$ system is equivalent to a rather simple system. But it must also be admitted that the other methods are probably more efficient, computationally.

Additional Exercises

To Be Written