39

Homogeneous Constant Matrix Systems, Part I

Finally, we can start discussing methods for solving a very important class of differential equation systems of differential equations: homogeneous constant matrix systems. These systems are important for at least three reasons:

1. We can develop straightforward procedures for solving them.
2. They arise in applications.
3. They provide approximations for more general systems and, because of this, will be instrumental later in analyzing the solutions to nonlinear autonomous systems.

Our goals for now are twofold:

1. To develop methods for finding the solutions to these systems.
2. To carefully examine the possible patterns for the trajectories of 2×2 constant matrix systems. (The results of this analysis will later be used when we study nonlinear systems.)

In this chapter we will concentrate on finding the "basic fundamental solutions", and analyzing those 2×2 systems for which these solutions suffice. Those cases that cannot be dealt with using just those "basic solutions" will be dealt with in the next chapter.

39.1 Basics, and Some Fundamental Solutions

We will now limit ourselves to homogeneous linear systems of the form

$$\mathbf{x}' = \mathbf{P}\mathbf{x}$$

in which the components of the $N \times N$ matrix $\mathbf{P}$ are all real-valued constants. To signify this, we will use $\mathbf{A}$ instead of $\mathbf{P}$ for this matrix, and refer to our system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ as a (basic) (real) *constant matrix* system.

So consider the system

$$\mathbf{x}' = \mathbf{A}\mathbf{x} \quad .$$

As already noted, the constant function

$$\mathbf{x}(t) = \mathbf{0} \qquad \text{for all} \quad t$$

is an equilibrium solution. And if $\mathbf{A}$ is invertible (i.e., $\det(\mathbf{A}) \neq 0$), then $\mathbf{x} = \mathbf{0}$ will be the only equilibrium solution.

What about the nonequilibrium solutions?

One approach to finding them is to generalize the solution to the above when $N = 1$ and the matrix $\mathbf{A}$ is just one real number A. Then our "system" is simply the first-order differential equation

$$x' = Ax \quad ,$$

whose solution is e^{At} multiplied by some arbitrary constant. But that approach requires figuring out just what we could mean by $e^{\mathbf{A}t}$ when $\mathbf{A}$ is a general $N \times N$ matrix. It is an interesting approach, but let's save that for later.

Instead, let us be inspired by the way we solved a single arbitrary N^{th}-order homogeneous ordinary differential equation

$$a_0 y^{(N)} + a_1 y^{(N-1)} + \cdots + a_{N-1} y' + a_N y = 0$$

back in chapters 16 and 18. There, the basic approach was to find a fundamental set of N solutions by initially assuming that each was of the form

$$y(x) = e^{rx}$$

where r is a value determined by plugging this formula for y into the differential equation and seeing what worked. If we were lucky, we found N different values for r — $r_1, r_2, \ldots, r_N$ — and could then write out our general solutions as a linear combination of these $e^{r_k x}$'s,

$$c_1 e^{r_1 x} + c_2 e^{r_2 x} + \cdots + c_N e^{r_N x} \quad .$$

Since we now know that a general solution to any homogeneous linear system can be constructed by a similar linear combination

$$c_1 \mathbf{x}^1(t) + c_2 \mathbf{x}^2(t) + \cdots + c_N \mathbf{x}^N(t)$$

where

$$\left\{ \mathbf{x}^1(t) , \mathbf{x}^2(t) , \ldots , \mathbf{x}^N(t) \right\}$$

is a fundamental set of solutions to our system $\mathbf{x}' = \mathbf{A}\mathbf{x}$, this seems to be a reasonable approach.

There is one slight issue: the $\mathbf{x}^k(t)$'s are vector-valued functions while the exponential e^{rt} is not. So let's try an exponential multiplied by some constant vector $\mathbf{u}$,

$$\mathbf{x}(t) = \mathbf{u}e^{rt} \quad .$$

Plugging this into our system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ and simplifying:

$$\frac{d}{dt}\left[\mathbf{u}e^{rt}\right] = \mathbf{A}\left[\mathbf{u}e^{rt}\right]$$

$$\hookrightarrow \qquad \mathbf{u}\, r e^{rt} = [\mathbf{A}\mathbf{u}]\, e^{rt}$$

$$\hookrightarrow \qquad \mathbf{u} r = \mathbf{A}\mathbf{u} \quad .$$

So, for $\mathbf{u}e^{rt}$ to be a solution to our system, the constant vector $\mathbf{u}$ and scalar r must satisfy

$$\mathbf{Au} = r\mathbf{u} \quad .$$

If we can find N pairs $(r_1, \mathbf{u}^1)$, $(r_2, \mathbf{u}^2)$, ... and $(r_N, \mathbf{u}^N)$ satisfying this equation and such that the Wronskian of

$$\left\{ \mathbf{u}^1 e^{r_1 t} \,,\, \mathbf{u}^2 e^{r_2 t} \,,\, \ldots \,,\, \mathbf{u}^N e^{r_N t} \right\}$$

is nonzero at one point (say, at $t = 0$) (which will certainly be the case if $\{\mathbf{u}^1, \mathbf{u}^2, \ldots, \mathbf{u}^N\}$ is a linearly independent set of column vectors), then we know this set is a fundamental set of solutions for $\mathbf{x}' = \mathbf{Ax}$, and that

$$\mathbf{x}(t) = c_1\mathbf{u}^1 e^{r_1 t} + c_2\mathbf{u}^2 e^{r_2 t} + \ldots + c_N\mathbf{u}^N e^{r_N t}$$

is a general solution.

So, our approach to actually solving the system of differential equations $\mathbf{x}' = \mathbf{Ax}$ begins with the attempt to find the pairs $(r, \mathbf{u})$ satisfying the algebraic equation $\mathbf{Au} = r\mathbf{u}$. Looks like it's time to review "eigenvectors and eigenvalues".

39.2 A Short Review of Eigenvalues and Eigenvectors (with Applications)

Eigen-Things

Let r be some scalar (i.e., a number, real or complex) and $\mathbf{u}$ some nonzero column vector. We say that r is an *eigenvalue* and $\mathbf{u}$ is a corresponding *eigenvector* for some given $N \times N$ matrix $\mathbf{A}$ if and only if

$$\mathbf{Au} = r\mathbf{u} \quad . \tag{39.1}$$

Alternatively, we can call r an eigenvalue corresponding to eigenvector $\mathbf{u}$. We can also simply say that $(r, \mathbf{u})$ is an *eigenpair* for the matrix $\mathbf{A}$.

Note that, while eigenvalues can be zero, we insist that eigenvectors be nonzero (because equation (39.1) reduces to the trivial equation $\mathbf{0} = \mathbf{0}$ if $\mathbf{u} = \mathbf{0}$ no matter what r is.)

!►*Example 39.1:* *Let*

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \quad , \quad \mathbf{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad , \quad \mathbf{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{w} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad .$$

Observe that

$$\mathbf{Au} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1+2 \\ 2+1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 3\mathbf{u} \quad .$$

and

$$\mathbf{Av} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1+2 \\ -2+1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = -1\begin{bmatrix} -1 \\ 1 \end{bmatrix} = (-1)\mathbf{v} \quad .$$

So $\mathbf{u} = [1, 1]^{\mathsf{T}}$ *is an eigenvector for* $\mathbf{A}$ *with corresponding eigenvalue* 3 *, and* $\mathbf{v} = [-1, 1]^{\mathsf{T}}$ *is an eigenvector for* $\mathbf{A}$ *with corresponding eigenvalue* -1 *. That is,*

$$\left(3, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) \qquad \text{and} \qquad \left(-1, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right)$$

are eigenpairs for the matrix $\mathbf{A}$ *.*

On the other hand,

$$\mathbf{Aw} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2+2 \\ 4+1 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix} \neq r\begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad \text{for any } r \quad .$$

So $\mathbf{w}$ *is not an eigenvector for* $\mathbf{A}$ *.*

As we discuss “eigen-things” it is worthwhile to keep in mind that we were led to this discussion by the discovery that $\mathbf{x}(t) = \mathbf{u}e^{rt}$ is a solution to the system $\mathbf{x}' = \mathbf{Ax}$ whenever $(r, \mathbf{u})$ satisfy $\mathbf{Au} = r\mathbf{u}$; that is, whenever $(r, \mathbf{u})$ is an eigenpair for $\mathbf{A}$. So let’s actually solve our first system of differential equations:

!►*Example 39.2:* *Consider the system*

$$\begin{aligned} x' &= x + 2y \\ y' &= 2x + y \end{aligned} \quad .$$

In matrix/vector form, this is

$$\mathbf{x}' = \mathbf{Ax}$$

where $\mathbf{A}$ *is the matrix from example 39.1. There we found two eigenpairs* $(3, \mathbf{u})$ *and* $(-1, \mathbf{v})$ *with*

$$\mathbf{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

So

$$\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-t} \right\} \qquad \left(\text{i.e.,} \quad \left\{ \begin{bmatrix} e^{3t} \\ e^{3t} \end{bmatrix}, \begin{bmatrix} -e^{-t} \\ e^{-t} \end{bmatrix} \right\} \right)$$

is a set of 2 *solutions to our* 2×2 *first-order system of differential equations. By inspection, it is clear that neither of these two solutions is a constant multiple of the other, so this set is clearly a linearly independent pair of solutions (if it’s not clear, compute the set’s Wronskian at* $t = 0$ *). Hence (invoking theorem 38.8), we know this set is a fundamental set of solutions for our system of differential equations, and that a general solution is given by*

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-t} \quad ,$$

or, equivalently, by

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} c_1 e^{3t} - c_2 e^{-t} \\ c_2 e^{3t} + c_2 e^{-t} \end{bmatrix} \quad .$$

It is important to note that eigenvectors are not unique. If $\mathbf{u}$ and $\mathbf{v}$ are both eigenvectors corresponding to the same eigenvalue r for $\mathbf{A}$, and α and β are any two scalars, then

$$\mathbf{A}[\alpha\mathbf{u} + \beta\mathbf{v}] = \alpha\mathbf{A}\mathbf{u} + \beta\mathbf{A}\mathbf{v} = \alpha[r\mathbf{u}] + \beta[r\mathbf{v}] = r[\alpha\mathbf{u} + \beta\mathbf{v}] \quad .$$

From this we get the following:

Lemma 39.1
Any nonzero linear combination of eigenvectors corresponding to a single eigenvalue r for a matrix $\mathbf{A}$ is also an eigenvector corresponding to eigenvalue r.

The set of all linear combinations of eigenvectors corresponding to a single eigenvalue r for $\mathbf{A}$ is, itself, a vector subspace of the set of all vectors. We will call this space the *eigenspace* corresponding to eigenvalue r for $\mathbf{A}$. The dimension of this space (i.e., the number of vectors in any basis for this eigenspace) is called the *geometric multiplicity* of eigenvalue r.

!►*Example 39.3:* *In an example just above, we saw that $\left(3, [1, 1]^{\mathsf{T}}\right)$ is an eigenpair for the 2×2 matrix in that example. Thus, for any nonzero constant c, the vector*

$$c\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}c\\c\end{bmatrix}$$

is also an eigenvector corresponding to eigenvalue 3. It turns out that there are no other eigenvectors corresponding to this eigenvalue, so the eigenspace corresponding to 3 is the vector space of all constant multiples of $[1, 1]^{\mathsf{T}}$ (including $0 \cdot [1, 1]^{\mathsf{T}} = [0, 0]^{\mathsf{T}}$). Clearly this is a one-dimensional vector space, so the geometric multiplicity of eigenvalue 3 is 1.

Finding (and Using) Eigen-Things

The key to finding each eigenpair $(r, \mathbf{u})$ lies in rewriting the basic defining equation $\mathbf{A}\mathbf{u} = r\mathbf{u}$ using the $N \times N$ identity matrix

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \quad .$$

Recall that $\mathbf{I}\mathbf{u} = \mathbf{u}$ for any column vector $\mathbf{u}$. So

$$\mathbf{A}\mathbf{u} = r\mathbf{u}$$

$$\hookrightarrow \qquad \mathbf{A}\mathbf{u} = r\mathbf{I}\mathbf{u}$$

$$\hookrightarrow \qquad \mathbf{A}\mathbf{u} - r\mathbf{I}\mathbf{u} = \mathbf{0}$$

$$\hookrightarrow \qquad [\mathbf{A} - r\mathbf{I}]\,\mathbf{u} = \mathbf{0} \quad . \tag{39.2}$$

Since the eigenvector $\mathbf{u}$ is required to be nonzero, the last equation can hold only if $\mathbf{A} - r\mathbf{I}$ is not invertible. Hence, we must have

$$\det[\mathbf{A} - r\mathbf{I}] = 0 \quad . \tag{39.3}$$

Computing this determinant yields an N^{th} degree polynomial equation with r as the unknown. This equation is called the *characteristic equation* for $\mathbf{A}$, and the polynomial is the called the *characteristic polynomial* for $\mathbf{A}$. From what we already know about such polynomial equations, we know that this equation will have N solutions — r_1, r_2, ..., r_N — some of which may be repeated.[1] The number of times a particular value of r is repeated in this set (which we normally call the 'multiplicity' of that value of r) will be called the *algebraic multiplicity* of that value of r so we don't confuse it with the geometric multiplicity of r.

This leads to the following procedure for finding all eigenpairs for an $N \times N$ matrix $\mathbf{A}$:

1. By computing the determinant, rewrite the characteristic equation,

$$\det[\mathbf{A} - r\mathbf{I}] = 0 \quad ,$$

as a polynomial equation.

2. Find all values of r — r_1, r_2, ..., r_M — satisfying this characteristic equation. Some of these may be repeated roots for the characteristic polynomial, and some may be complex.

3. For each different r_k, set

$$\mathbf{u}^k = \begin{bmatrix} u_1^k \\ u_2^k \\ \vdots \\ u_N^k \end{bmatrix}$$

and solve either

$$\mathbf{A}\mathbf{u}^k = r_k\mathbf{u}^k \qquad \text{or} \qquad [\mathbf{A} - r\mathbf{I}]\,\mathbf{u}^k = \mathbf{0}$$

for all possible u_j^k's.

Two practical notes on this:

(a) Don't really use the notation "u_j^k". Use whatever notation for the components of $\mathbf{u}^k$ seems convenient.

(b) Since there is a whole space of eigenvectors corresponding to each eigenvalue, you will not obtain "an eigenvector $\mathbf{u}^k$". Instead, you will obtain a description of all the eigenvectors in this eigenspace in terms of a minimal set of arbitrary constants.

There is at least one more step we will want to carry out before using our results to solve our system of differential equations, $\mathbf{x}' = \mathbf{A}\mathbf{x}$, but maybe we had better do an example, first.

!► *Example 39.4:* *Let us find the eigenvalues and corresponding eigenvectors for*

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & -5 \\ 0 & 6 & 0 \\ 1 & 0 & 7 \end{bmatrix} \quad .$$

[1] See section 18.1 on page 429 for a discussion of N^{th} degree polynomial equations.

First, we reduce the characteristic equation to polynomial form:

$$\det[\mathbf{A} - r\mathbf{I}] = 0$$

$$\hookrightarrow \quad \det\left(\begin{bmatrix}1 & 0 & -5\\0 & 6 & 0\\1 & 0 & 7\end{bmatrix} - r\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\right) = 0$$

$$\hookrightarrow \quad \det\left(\begin{bmatrix}1 & 0 & -5\\0 & 6 & 0\\1 & 0 & 7\end{bmatrix} - \begin{bmatrix}r & 0 & 0\\0 & r & 0\\0 & 0 & r\end{bmatrix}\right) = 0$$

$$\hookrightarrow \quad \det\begin{bmatrix}1-r & 0 & -5\\0 & 6-r & 0\\1 & 0 & 7-r\end{bmatrix} = 0$$

$$\hookrightarrow \quad (1-r)(6-r)(7-r) + 5(6-r) = 0$$

$$\hookrightarrow \quad (6-r)[(1-r)(7-r) + 5] = 0$$

$$\hookrightarrow \quad (6-r)\left[r^2 - 8r + 12\right] = 0 \quad .$$

Note that we did not completely multiply out the terms. Instead, we factored out the common factor $6 - r$ *to simplify the next step, which is to find all solutions to the characteristic equation. Fortunately for us, we can easily factor the rest of the polynomial, obtaining*

$$(6-r)\left[r^2 - 8r + 12\right] = (6-r)[(r-2)(r-6)] = -(r-2)(r-6)^2$$

as the completely factored form for our characteristic polynomial. Thus our characteristic equation $\det[\mathbf{A} - r\mathbf{I}]$ *reduces to*

$$-(r-2)(r-6)^2 = 0 \quad ,$$

which we can solve by inspection. We have two eigenvalues

$$r = 2 \qquad \text{and} \qquad r = 6$$

(with $r = 6$ *having algebraic multiplicity* 2 *).*

To find the eigenvectors corresponding to eigenvalue $r = 2$ *, we set* $r = 2$ *and solve*

$$[\mathbf{A} - r\mathbf{I}]\mathbf{u} = \mathbf{0}$$

for the components of $\mathbf{u}$ *. For convenience, let* $\mathbf{u} = [\alpha, \beta, \gamma]^{\mathsf{T}}$ *. Then*

$$[\mathbf{A} - r\mathbf{I}]\,\mathbf{u} = \mathbf{0}$$

$$\hookrightarrow \quad \left(\begin{bmatrix}1 & 0 & -5\\0 & 6 & 0\\1 & 0 & 7\end{bmatrix} - 2\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\right)\begin{bmatrix}\alpha\\\beta\\\gamma\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\hookrightarrow \quad \begin{bmatrix} 1-2 & 0 & -5 \\ 0 & 6-2 & 0 \\ 1 & 0 & 7-2 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\hookrightarrow \quad \begin{bmatrix} -1 & 0 & -5 \\ 0 & 4 & 0 \\ 1 & 0 & 5 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad .$$

So the unknowns, α *,* β *and* γ *must satisfy the algebraic system*

$$\begin{aligned} -\alpha + 0\beta - 5\gamma &= 0 \\ 0\alpha + 4\beta + 0\gamma &= 0 \\ 1\alpha + 0\beta + 5\gamma &= 0 \end{aligned} \quad .$$

Using whichever method you like, this is easily reduced to

$$\beta = 0 \qquad \text{and} \qquad \alpha + 5\gamma = 0 \quad .$$

We can choose either α *or* γ *to be an arbitrary constant. Choosing* γ *, we then must have* $\alpha = -5\gamma$ *to satisfy the last equation above. Thus, the eigenvectors corresponding to eigenvalue* $r = 2$ *are given by*

$$\mathbf{u} = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} -5\gamma \\ 0 \\ \gamma \end{bmatrix} = \gamma \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix}$$

where γ *can be any nonzero value.*

To find the eigenvectors corresponding to eigenvalue $r = 6$ *, we set* $r = 6$ *and solve*

$$[\mathbf{A} - r\mathbf{I}]\mathbf{u} = \mathbf{0}$$

for the components of $\mathbf{u}$ *. For convenience, let us again use the notation*

$$\mathbf{u} = [\alpha , \beta , \gamma]^{\mathsf{T}} \quad .$$

Then

$$[\mathbf{A} - r\mathbf{I}]\,\mathbf{u} = \mathbf{0}$$

$$\hookrightarrow \quad \left(\begin{bmatrix} 1 & 0 & -5 \\ 0 & 6 & 0 \\ 1 & 0 & 7 \end{bmatrix} - 6 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\hookrightarrow \quad \begin{bmatrix} -5 & 0 & -5 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad ,$$

which reduces to

$$\alpha + \gamma = 0 \qquad \text{and} \qquad 0\beta = 0 \quad .$$

In this case let us take α to be an arbitrary constant with $\gamma = -\alpha$ so that the first equation is satisfied. The second equation is satisfied no matter what β is, so β is another arbitrary constant. Thus, the eigenvectors corresponding to eigenvalue $r = 2$ are given by

$$\mathbf{u} = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \\ -\alpha \end{bmatrix} = \begin{bmatrix} \alpha \\ 0 \\ -\alpha \end{bmatrix} + \begin{bmatrix} 0 \\ \beta \\ 0 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

where α and β are arbitrary constants.

For our applications, we do not want a description of all possible eigenvectors corresponding to each eigenvalue r . Instead we want a set of specific eigenvectors

$$\{ \mathbf{u}^1 , \mathbf{u}^2 , \ldots , \mathbf{u}^M \}$$

such that all the eigenvectors in this set corresponding to any one particular eigenvalue form a basis for that eigenvalue's eigenspace. So, the last step in our procedure for finding the eigenvalues and eigenvectors for our matrix is:

4. For each distinct eigenvalue, choose appropriate values for the arbitrary constants describing the corresponding eigenvectors so as to obtain specific vectors for a basis for each eigenspace. Then write down the collection of all the eigenvectors so found,

$$\{ \mathbf{u}^1 , \mathbf{u}^2 , \ldots , \mathbf{u}^M \} \quad ,$$

keeping track of their corresponding eigenvalues.

By the way, recall that the number of vectors in any basis for an eigenspace corresponding to eigenvalue r (i.e., the dimension of that eigenspace) is called the geometric multiplicity of that eigenvalue. With a little thought, you will realize that

$$\begin{aligned} &\text{geometric multiplicity of eigenvalue } r \\ &\qquad = \text{number of arbitrary constants needed to describe all corresponding eigenvectors.} \end{aligned}$$

Along these lines, we might as well mention that, for any eigenvalue r_k ,

$$1 \leq \text{geometric multiplicity of } r_k \leq \text{algebraic multiplicity of } r_k \quad .$$

Let us just accept this fact as something usually verified in a course on linear algebra.

!► *Example 39.5:* *In the last example, we saw that*

$$\mathbf{u} = \gamma \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix}$$

describes all the eigenvectors corresponding to eigenvalue $r = 2$ for the matrix $\mathbf{A}$ from example 39.4. Since we only have one arbitrary constant, γ , the geometric multiplicity for $r = 2$ is 1 . Setting $\gamma = 1$ gives us the one eigenvector we need corresponding to $r = 2$,

$$\mathbf{u}^1 = \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix} \quad .$$

(Any other nonzero value for z would also have given us a suitable eigenvector.)

We also saw that the eigenvectors corresponding to eigenvalue $r = 6$ for $\mathbf{A}$ are all given by

$$\alpha \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

In this case, we have 2 arbitrary constants, so the geometric multiplicity is 2. Taking $\alpha = 1$ and $\beta = 0$ gives us the eigenvector

$$\mathbf{u}^2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \quad ,$$

and taking $\alpha = 0$ with $\beta = 1$ gives us the eigenvector

$$\mathbf{u}^3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad .$$

Clearly, these two eigenvectors form a linearly independent pair.

In summary, $\{\mathbf{u}^1, \mathbf{u}^2, \mathbf{u}^3\}$ is our set of eigenvectors with $\mathbf{u}^1$ corresponding to $r = 2$, and $\mathbf{u}^2$ and $\mathbf{u}^3$ both corresponding to $r = 6$.

Let's not forget why we are finding these eigenvalues and eigenvectors. We want to use them to construct solutions to systems of differential equations.

!►*Example 39.6:* *Consider the 3×3 system of differential equations*

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 1 & 0 & -5 \\ 0 & 6 & 0 \\ 1 & 0 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad .$$

From examples 39.4 and 39.5, above, we know that the matrix in this equation has eigenvalues

$$r_1 = 2 \quad , \quad r_2 = 6 \quad \text{and} \quad r_3 = 6 \quad ,$$

and corresponding eigenvectors

$$\mathbf{u}^1 = \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix} \quad , \quad \mathbf{u}^2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{u}^3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad ,$$

respectively. And from our discussion in the previous section, we know this gives the corresponding set of 3 solutions to our 3×3 system of differential equations

$$\left\{ \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix} e^{2t} , \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} e^{6t} , \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^{6t} \right\} \quad .$$

Checking the Wronskian at $t = 0$, we get

$$W(0) = \det \begin{bmatrix} -5 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix} = 6 \neq 0 \quad ,$$

assuring us (via theorem 38.9) that our set of solutions is a fundamental set, and that a general solution to our system is given by

$$\begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = c_1 \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} e^{6t} + c_3 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^{6t} \quad ,$$

or, equivalently, by

$$\begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} -5c_1 e^{2t} - c_2 e^{6t} \\ c_3 e^{6t} \\ c_1 e^{2t} + c_2 e^{6t} \end{bmatrix} \quad .$$

Do note that, had we chosen different values for the 'arbitrary' constants in exercise 39.5, then the specific vectors in our general solution would be different. However, the end result would have been equivalent to the above.

Notes on Finding and Using Eigenpairs
Complete and Incomplete Sets of Eigenvectors

From what we (should) know about linear algebra, we are assured that the procedure just described will generate a linearly independent set of eigenvectors

$$\left\{ \mathbf{u}^1 , \mathbf{u}^2 , \ldots , \mathbf{u}^M \right\}$$

for our $N \times N$ real constant matrix $\mathbf{A}$, with the size M of this set being no larger than N. Moreover:

1. M will not depend on how we choose the $\mathbf{u}^k$'s, but only on the matrix $\mathbf{A}$.
2. The components of $\mathbf{u}^k$ can be chosen as real numbers provided its corresponding eigenvalue is real.

If we are lucky, $M = N$. Then we will have a linearly independent set of N eigenvectors

$$\left\{ \mathbf{u}^1 , \mathbf{u}^2 , \ldots , \mathbf{u}^N \right\}$$

corresponding, respectively, to an ordered set of N eigenvalues

$$\{ r_1 , r_2 , \ldots , r_N \}$$

(with some of the eigenvalues, possibly, being repeated). If this happens, we say that the matrix has a *complete set of eigenvectors*, with any set of eigenvectors being called a *complete set of eigenvectors* if and only if it is a linearly independent set of N eigenvectors for $\mathbf{A}$.

We like for our set of eigenvectors to be complete, because then

$$\left\{ \mathbf{u}^1 e^{r_1 t} , \mathbf{u}^2 e^{r_2 t} , \ldots , \mathbf{u}^N e^{r_N t} \right\}$$

is a set of N solutions to our $N \times N$ constant matrix system $\mathbf{x}' = \mathbf{A}\mathbf{x}$. Moreover, it is a linearly independent set of vectors when $t = 0$. That means the Wronskian for this set is nonzero, which, in turn means that this set is a fundamental set of solutions for our system of differential equations, and, thus,

$$\mathbf{x}(t) = c_1\mathbf{u}^1 e^{r_1 t} + c_2\mathbf{u}^2 e^{r_2 t} + \cdots + c_N\mathbf{u}^N e^{r_1 t}$$

is a general solution to $\mathbf{x}' = \mathbf{A}\mathbf{x}$. That was the situation in example 39.6, above.

There are two cases where we are assured that a given $N \times N$ matrix $\mathbf{A}$ has a complete set of eigenvectors before we begin seeking these eigenvectors:

1. When the matrix has N different eigenvalues.

2. When the matrix is symmetric (i.e., $\mathbf{A}^\mathsf{T} = \mathbf{A}$).

Verifying the first is easy: Since each of the N eigenvalues must have at least one corresponding eigenvector, the process above will generate a linearly independent set of at least N eigenvectors. But, as noted above, this set can contain at most N eigenvectors. Hence, it must contain exactly N eigenvectors. Moreover, it follows with just a little work that, for each eigenvector r_k,

$$1 = \text{geometric multiplicity of } r_k = \text{algebraic multiplicity of } r_k \quad .$$

Verifying that we have a complete set of eigenvectors when the matrix is symmetric is a bit harder to show. Fortunately for us, you almost certainly discussed this in a course on linear algebra since the following is a standard part of most introductory algebra courses:

Theorem 39.2
Let $\mathbf{A}$ be a symmetric $N \times N$ matrix (with real-valued components). Then both of the following hold:

1. *All the eigenvalues are real.*

2. *There is an orthogonal basis for $\mathbb{R}^N$ consisting of eigenvectors for $\mathbf{A}$.*

Shortcuts with Triangular Matrices

Given a matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1N} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2N} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{N1} & a_{N2} & a_{N3} & \cdots & a_{NN} \end{bmatrix} \quad ,$$

the *main diagonal* is ‘diagonal set’ of entries from the upper left corner to the lower right corner of the matrix. That is, it is the diagonal consisting of a_{11}, a_{22}, $\ldots$ and a_{NN}.

We say that our matrix $\mathbf{A}$ is *upper triangular* or *lower triangular* if, respectively, all the entries below or all the entries above the main diagonal are zero,

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1N} \\ 0 & a_{22} & a_{23} & \cdots & a_{2N} \\ 0 & 0 & a_{33} & \cdots & a_{3N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{NN} \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ a_{31} & a_{32} & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{N1} & a_{N2} & a_{N3} & \cdots & a_{NN} \end{bmatrix} \quad .$$

And if $\mathbf{A}$ is both upper and lower triangular; that is, if all the entries off of the main diagonal are zero,

$$\begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \cdots & 0 \\ 0 & 0 & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{NN} \end{bmatrix} \quad ,$$

then we say that the matrix $\mathbf{A}$ is *diagonal*.

When $\mathbf{A}$ is any of these forms, you can easily use the rules for computing determinants to derive

$$\det(\mathbf{A} - \lambda\mathbf{I}) \; = \; (a_{11} - \lambda)(a_{22} - \lambda)(a_{33} - \lambda) \cdots (a_{NN} - \lambda) \quad ,$$

And so the characteristic equation $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$ immediately reduces

$$(a_{11} - \lambda)(a_{22} - \lambda)(a_{33} - \lambda) \cdots (a_{NN} - \lambda) \; = \; 0$$

telling us that the elements of this matrix's main diagonal are the eigenvalues of any upper or lower triangular matrix.

What's more, if $\mathbf{A}$ is diagonal, then, virtually by inspection, you can see that an eigenvector $\mathbf{u}^k$ corresponding to $\lambda = a_{kk}$ is the column vector whose components are all 0 except for the k^{th} component, which we can take to be 1.

So if you are dealing with a upper or lower triangular matrix, you can simply read off the eigenvalues from the main diagonal. And if the matrix is diagonal, you can also immediately give the corresponding eigenvectors.

39.3 Eigenpairs and Corresponding Solutions

Sets of Solutions from Sets of Eigenpairs

As just discussed, our method will generate a set of eigenpairs

$$\left\{ (r_1, \mathbf{u}^1) \, , \; (r_2, \mathbf{u}^2) \, , \; \ldots \, , \; (r_M, \mathbf{u}^M) \right\}$$

for our constant $N \times N$ matrix $\mathbf{A}$, with the set

$$\left\{ \mathbf{u}^1 \, , \; \mathbf{u}^2 \, , \; \ldots \, , \; \mathbf{u}^M \right\}$$

being a linearly independent set of column vectors. Moreover, this method will find the largest possible such set of eigenvectors. Any other eigenvector will be a linear combination of vectors from the above set.

Our interest in these sets came from the observations made in section 39.1 that can be summarized, using terminology from the last section, as the following lemma and theorem:

Lemma 39.3

Let

$$\left\{ \mathbf{u}^1 \, , \; \mathbf{u}^2 \, , \; \ldots \, , \; \mathbf{u}^M \right\}$$

be a linearly independent set of eigenvectors for a constant $N \times N$ *matrix* $\mathbf{A}$*, and, for each* $\mathbf{u}^k$*, let* r_k *be the corresponding eigenvalue. Then*

$$\left\{ \mathbf{u}^1 e^{r_1 t} , \mathbf{u}^2 e^{r_2 t} , \ldots , \mathbf{u}^M e^{r_M t} \right\}$$

is a set of solutions for the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ *on the entire real line. Moreover, this is a linearly independent set of vectors when* $t = 0$*.*

Theorem 39.4
Let

$$\left\{ \mathbf{u}^1 , \mathbf{u}^2 , \ldots , \mathbf{u}^N \right\}$$

be a complete set of eigenvectors for a constant $N \times N$ *matrix* $\mathbf{A}$*, and, for each* $\mathbf{u}^k$*, let* r_k *be the corresponding eigenvalue. Then*

$$\left\{ \mathbf{u}^1 e^{r_1 t} , \mathbf{u}^2 e^{r_2 t} , \ldots , \mathbf{u}^N e^{r_N t} \right\}$$

is a fundamental set of solutions for the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ *on the entire real line, and*

$$\mathbf{x}(t) = c_1 \mathbf{u}^1 e^{r_1 t} + c_2 \mathbf{u}^2 e^{r_2 t} + \cdots + c_N \mathbf{u}^N e^{r_1 t}$$

is a general solution to $\mathbf{x}' = \mathbf{A}\mathbf{x}$*.*

Consequently, if the matrix $\mathbf{A}$ has a complete set of eigenvectors, the method discussed in the last section, along with the observations made in the last theorem, will lead to a general solution to $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

Issues

We are still left with the issue of just what to do if our matrix $\mathbf{A}$ does not have a complete set of eigenvectors. We can still find a linearly independent set of eigenvectors,

$$\left\{ \mathbf{u}^1 , \mathbf{u}^2 , \ldots , \mathbf{u}^M \right\} \quad ,$$

and

$$\left\{ \mathbf{u}^1 e^{r_1 t} , \mathbf{u}^2 e^{r_2 t} , \ldots , \mathbf{u}^M e^{r_M t} \right\}$$

is still a set of solutions to $\mathbf{x}' = \mathbf{A}\mathbf{x}$, but it is not a fundamental set. We will need to discover how to find additional solutions to fill out this set to a 'complete' fundamental set of N solutions. We will do this in the next chapter.

There is another little issue that we have sidestepped: the fact that the eigenvalues and the components of the eigenvectors can be complex values. We will also deal with that issue in the next chapter.

For now, let us take what we have and examine the possible solutions and their trajectories, with particular emphasis on the trajectories of solutions to 2×2 systems. We will find that this will lead to some very interesting pictures and results that will even be useful when dealing with more general systems of differential equations.

39.4 Two-Dimensional Phase Portraits: Preliminaries

It turns out that the eigenvalues and eigenvectors tell us much about the qualitative behavior of the solutions to a constant matrix system $\mathbf{x}' = \mathbf{A}\mathbf{x}$. To see this most readily, we will carefully determine the possible trajectories when $\mathbf{A}$ is 2×2. A major goal is to determine how the eigenvalues and eigenvectors determine the possible "patterns" of the trajectories in the phase portraits for $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

The Critical Point at the Origin

Remember that the origin $(0, 0)$ is a critical point for any system $\mathbf{x}' = \mathbf{A}\mathbf{x}$, and that this single point is the entire trajectory of the equilibrium solution

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \text{for all} \quad t \quad .$$

If $\det \mathbf{A} \neq 0$ then the origin is the only critical point, and $\mathbf{x}(t) = \mathbf{0}$ is the only equilibrium solution.

Much of our work in what follows will be to determine the stability of this critical point, and to determine "geometries" of the trajectories in regions both close to $(0, 0)$ and in regions far away from $(0, 0)$. In the process we will introduce terminology for further classifying the critical point $(0, 0)$ according to these "geometries". Some of the terms we will introduce are "nodes", "saddle points", "centers" and "spiral points". (However, we won't fully define these terms until later, in section 41.3.)

Symmetries

The sketching of a phase portrait for $\mathbf{x}' = \mathbf{A}\mathbf{x}$ can be aided by noting simple symmetries inherent in the direction field of a basic constant matrix system. To see this, let (x_0, y_0) be any point in the plane other that the origin, and let $\mathbf{x}^0 = [x_0, y_0]^\mathsf{T}$. Remember, the direction arrow for our system at (x_0, y_0) is a short arrow drawn at this point in the same direction as $\mathbf{v}^0$ where

$$\mathbf{v}^0 = \mathbf{A}\mathbf{x}^0 \quad .$$

Now consider the direction arrow at any nonzero point (x_1, y_1) on the straight half line starting at the origin and going through (x_0, y_0). Then, for some $c > 0$

$$(x_1, y_1) = (cx_0, cy_0) \qquad , \qquad \mathbf{x}^1 = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} cx_0 \\ cy_0 \end{bmatrix} = c\mathbf{x}^0$$

and the direction arrow for our system at (x_1, y_1) is a short arrow drawn at this point in the same direction as

$$\mathbf{A}\mathbf{x}^1 = \mathbf{A}c\mathbf{x}^0 = c\mathbf{A}\mathbf{x}^0 = c\mathbf{v}_0 \quad .$$

But multiplying a vector by a positive constant does not change its direction. So, as illustrated in figure 39.1a,

> *The direction arrow at each point on a half line having the origin as an endpoint all point in exactly the same direction (or are all* $\mathbf{0}$ *).*

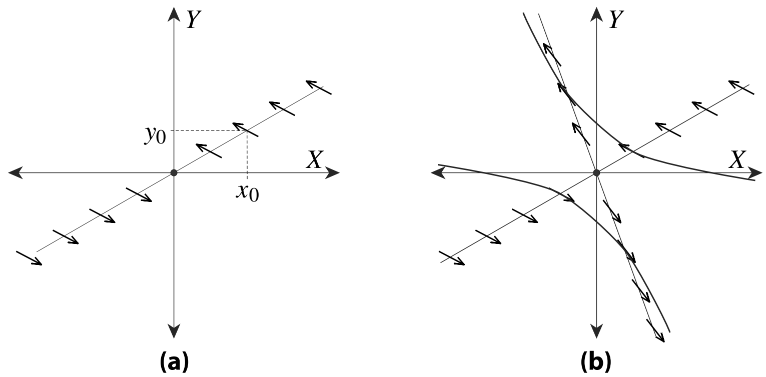


Figure 39.1: Symmetries in direction fields and phase portraits for a 2×2 constant matrix system.

On the other hand, the direction arrow at $(-x_0, -y_0)$ is in the same direction as

$$\mathbf{A}\begin{bmatrix}-x_0\\-y_0\end{bmatrix} = -\mathbf{A}\mathbf{x}^0 = -\mathbf{v} \quad .$$

So, as also illustrated in figure 39.1a,

> *The direction arrow at* $(-x_0, -y_0)$ *is in the exact opposite direction as is the direction arrow at* (x_0, y_0) *.*

In other words, the direction field is antisymmetric across the origin.

Using the above observations, we can sketch a useful direction field after computing the direction arrows at only a few well-chosen points. Moreover, these patterns in the direction field for must be carried over in the corresponding phase portrait, with, say the trajectories in the lower-half plane being "mirror images reflected across the origin" of the trajectories in the upper half plane, as illustrated in figure 39.1b.

39.5 Two-Dimensional Phase Portraits from Real, Complete Eigen-Sets

Throughout this section, we will assume $\mathbf{A}$ is a 2×2 constant matrix with real components and a complete set of eigenvectors $\{\mathbf{u}^1, \mathbf{u}^2\}$. We will further assume that the corresponding eigenvalues r_1 and r_2 are real and nonzero.[2] That means our solutions will be of the form

$$\mathbf{x}(t) = c_1\mathbf{u}^1 e^{r_1 t} + \mathbf{u}^2 e^{r_2 t}$$

where each component of each $\mathbf{u}^k$ is a real number.

As t varies, each solution traces out a trajectory in the plane. We want to discover the possible patterns of the trajectories, and how these patterns depend on the eigenvalues and eigenvectors.

[2] You'll deal with zero eigenvalues in exercise 39.13.

In fact, we will discover that the eigenpairs of $\mathbf{A}$, $(r_1, \mathbf{u}^1)$ and $(r_2, \mathbf{u}^2)$, alone, give us enough information to sketch crude, yet enlightening, phase portraits for $\mathbf{x}' = \mathbf{A}\mathbf{x}$. If we then wish, we can refine our sketches of the phase portraits by using minimal direction fields and the symmetries discussed in the previous section.[3]

For convenience, let us use the notation

$$\mathbf{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \quad , \quad \mathbf{u}^1 = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \quad \text{and} \quad \mathbf{u}^2 = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \quad ,$$

so that our solutions will look like

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} e^{r_1 t} + c_2 \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} e^{r_2 t} \quad ,$$

and our trajectories will be curves on the XY–plane. Let us also assume we've labeled our eigenvalues so that $r_1 \leq r_2$.

Trajectories Corresponding to One Eigenpair

We should first consider the trajectory of a single solution of the form

$$\mathbf{x}(t) = c\mathbf{u}e^{rt}$$

where c is any nonzero constant, and $(r, \mathbf{u})$ is any eigenpair with r being real and nonzero.

!►*Example 39.7:* *Suppose* $\mathbf{x}(t) = c\mathbf{u}e^{rt}$ *with*

$$(r\ ,\ \mathbf{u}) = \left(3\,, \begin{bmatrix} 4 \\ 1 \end{bmatrix}\right)$$

and c *is any real number. Then*

$$\mathbf{x}(t) = c\mathbf{u}e^{3t} = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c \begin{bmatrix} 4 \\ 1 \end{bmatrix} e^{3t} \quad ,$$

which means $\mathbf{x}(t)$ *is just some multiple of* $\mathbf{u}$ *for each* t*, and that, in turn, means that, for each real value* t*,* $(x(t), y(t))$ *is a point on the line through the origin and parallel to* $\mathbf{u}$*, as illustrated in figures 39.2a and 39.2b. Furthermore:*

1. *If* $c = 0$*, then our solution reduces to the already-known equilibrium solution* $\mathbf{x}(t) = \mathbf{0}$*, whose entire trajectory is just the critical point* $(0, 0)$*.*

2. *If* $c > 0$ *and* t *is any real number, then* $c\mathbf{u}e^{3t}$ *is a positive multiple of* $\mathbf{u}$*, which means* $(x(t), y(t))$ *is always on the half line extending from* $(0, 0)$ *in the direction of* $\mathbf{u}$*, as in figure 39.2a. Moreover, since* e^{3t} *is a continuous increasing function on the real line, and*

$$\lim_{t\to-\infty} (x(t), y(t)) = \lim_{t\to-\infty} \left(c \cdot 3e^{3t}, c \cdot 2e^{3t}\right) = (0, 0) \quad ,$$

[3] Of course, you can also use a suitable computer math package to sketch phase portraits, as the author has done for this text. But we will find the knowledge and experience gained by "hand sketching" phase portraits invaluable later on.

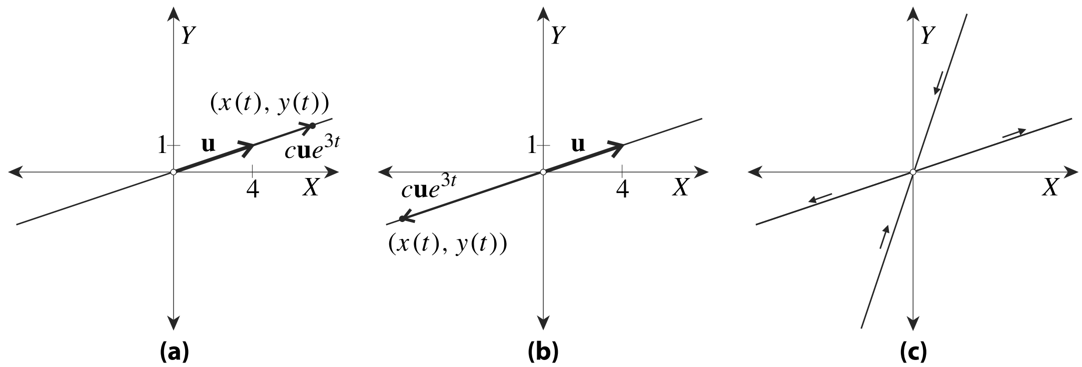


Figure 39.2: Plotting the trajectory of $c\mathbf{u}e^{3t}$ with $\mathbf{u} = [4, 1]^\mathsf{T}$ **(a)** when $c > 0$ and **(b)** when $c < 0$. **(c)** Trajectories corresponding to eigenpairs $(3, [4, 1]^\mathsf{T})$ and $(-3, [1, 4]^\mathsf{T})$.

while

$$\lim_{t \to +\infty} (x(t), y(t)) = \lim_{t \to +\infty} \left(c \cdot 3e^{3t}, c \cdot 2e^{3t}\right) = \text{“}(+\infty, +\infty)\text{”} \quad ,$$

it should be clear that, in fact, $(x(t), y(t))$ traces out this entire half line with

$$(x(t), y(t)) \text{ going from } (0, 0) \text{ to } (+\infty, +\infty)$$

as

$$t \text{ goes from } -\infty \text{ to } +\infty \quad .$$

So the trajectory is this entire half line parallel to $\mathbf{u}$ with endpoint $(0, 0)$, and with the direction of travel being the direction away from the origin. (Strictly speaking, the origin is not part of the trajectory since t can only approach $-\infty$, never equal $-\infty$.)

3. *If $c < 0$ and t is any real number, then $c\mathbf{u}e^{3t}$ is a negative multiple of $\mathbf{u}$, and, for reasons very similar to those given when $c > 0$, it should be clear that the trajectory is the half line extending from $(0, 0)$ in the direction of $-\mathbf{u}$, and with the direction of travel being in the direction away from the origin (as illustrated in figure 39.2b). (Again, while $(0, 0)$ is an endpoint for this trajectory, it is not, strictly speaking, in the trajectory.)*

!► *Example 39.8:* *Let us now consider the trajectories for*

$$\mathbf{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c \begin{bmatrix} 1 \\ 4 \end{bmatrix} e^{-3t}$$

where c is any real number.

The general ideas given in the previous example still apply, and lead us to the trajectories all being parts of the straight line through the origin and parallel to the vector $\mathbf{u}$. Again, if $c = 0$, then the trajectory is just the critical point $(0, 0)$; if $c > 0$, then the trajectory is the entire half line with endpoint $(0, 0)$ and extending in the direction of $\mathbf{u}$; if $c < 0$, then the trajectory is the entire half line with endpoint $(0, 0)$ and extending in the direction of $-\mathbf{u}$. This time, however,

$$\lim_{t \to +\infty} (x(t), y(t)) = \lim_{t \to +\infty} \left(c \cdot 1e^{-3t}, c \cdot 4e^{-3t}\right) = (0, 0) \quad .$$

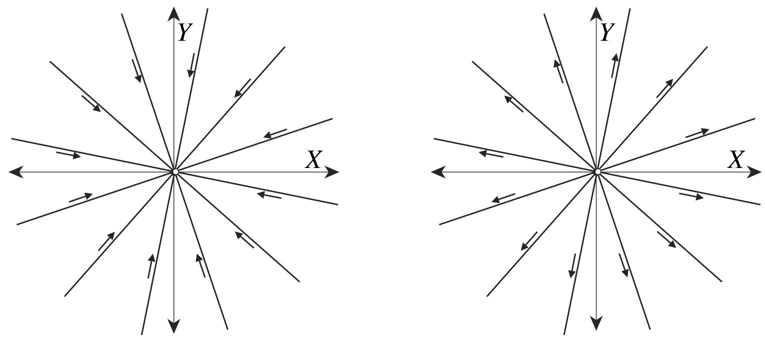


Figure 39.3: Phase portraits exhibiting star nodes with **(a)** the eigenvalue being negative and **(b)** the eigenvalue being positive.

So $(x(t), y(t))$ approaches $(0,0)$ as t increases. That is, the direction of travel on these trajectories are towards the origin, as illustrated in figure 39.2c.

Clearly, the discussion carried out for these two examples can be carried out in general, and will lead to the observation that, if r is any nonzero real number, then the trajectories for any nonequilibrium solution of the form $\mathbf{x}(t) = c\mathbf{u}e^{rt}$ are the two straight half lines "starting" at the origin and parallel to the vector $\mathbf{u}$. The direction of travel will be away from the origin if $r > 0$ and towards the origin if $r < 0$.

For simplicity, let us call these the *straight-line trajectories* corresponding to the eigenpair $(r, \mathbf{u})$ (assuming $(r, \mathbf{u})$ is an eigenpair for the matrix in our system of differential equations). Sketching these will usually be one of the first things we will want to do when sketching phase portraits. (Remember, even though we are calling these "straight-line" trajectories, they are really "straight half-line" trajectories. Each has $(0,0)$ as an endpoint, though, strictly speaking, $(0,0)$ is not a point in these trajectories.)

Trajectories When $r_1 = r_2 \neq 0$

Suppose the matrix in our 2×2 system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ has a single repeated nonzero eigenvalue and a complete set of eigenvectors $\{\mathbf{u}^1, \mathbf{u}^2\}$. Then $r_1 = r_2$, and

$$\mathbf{x}(t) = c_1\mathbf{u}^1 e^{r_1 t} + c_2\mathbf{u}^2 e^{r_2 t} = \left[c_1\mathbf{u}^1 + c_2\mathbf{u}^2\right] e^{r_1 t} \quad .$$

But since $\{\mathbf{u}^1, \mathbf{u}^2\}$ is a complete set of eigenvectors, $c_1\mathbf{u}^1 + c_2\mathbf{u}^2$ describes all 2-dimensional vectors. So the above general solution simplifies to

$$\mathbf{x}(t) = \mathbf{u}e^{r_1 t}$$

where $\mathbf{u}$ is an arbitrary column vector. Consequently, the set of all trajectories of this system is the set of all half lines in the plane with endpoints at $(0,0)$, with the direction of travel being away from the origin if $r_1 > 0$, and towards the origin if $r_1 < 0$ (as illustrated in figure 39.3).

In cases like this, the critical point (the trajectory for the equilibrium solution) is said to be a "node" for our system $\mathbf{x}' = \mathbf{A}\mathbf{x}$. Sometimes, such a point is also called a "proper node" or even a "star node" because of the way the trajectories seem to radiate from the origin. If the eigenvalues are negative, then this critical point and corresponding equilibrium solution $\mathbf{x}(t) = \mathbf{0}$

are asymptotically stable (see figure 39.3a). If the eigenvalues are positive, then this critical point and equilibrium solution are unstable (see figure 39.3b).

By the way, in exercise 39.10, you will show that any 2×2 matrix having a complete set of eigenvectors all corresponding to the same eigenvalue r is a very simple diagonal matrix.

Trajectories When $0 < r_1 < r_2$

We start with an illustrative example.

!►*Example 39.9:* *Consider a 2×2 system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ whose matrix has eigenpairs*

$$(r_1, \mathbf{u}^1) = \left(2, \begin{bmatrix} 4 \\ 1 \end{bmatrix}\right) \quad \text{and} \quad (r_2, \mathbf{u}^2) = \left(5, \begin{bmatrix} 1 \\ 4 \end{bmatrix}\right) \quad .$$

The general solution is then

$$\mathbf{x}(t) = c_1 \mathbf{u}^1 e^{r_1 t} + c_2 \mathbf{u}^2 e^{r_2 t} = c_1 \begin{bmatrix} 4 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ 4 \end{bmatrix} e^{5t} \quad . \tag{39.4}$$

Observe that

$$\lim_{t \to -\infty} \mathbf{x}(t) = \lim_{t \to -\infty} c_1 \begin{bmatrix} 4 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ 4 \end{bmatrix} e^{5t} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad .$$

This tells us that the trajectories all "start" at the origin. The fact that both e^{2t} and e^{5t} become infinite as $t \to +\infty$ tells us that all nonequilibrium trajectories extend infinitely out on the plane.

We start our sketch by plotting a dot at the origin for the equilibrium solution, and sketching the straight-line trajectories corresponding to the eigenpairs $(r_1, \mathbf{u}^1)$ and $(r_2, \mathbf{u}^2)$. Note that the direction of travel on each of these straight-line trajectories is away from the origin. This takes care of the trajectories of solution (39.4) when either $c_1 = 0$ or $c_2 = 0$.

To sketch any other trajectory, assume c_1 and c_2 are both nonzero, and consider some convenient tangent vectors using the the derivative of $\mathbf{x}(t)$,

$$\mathbf{x}'(t) = 2c_1 \mathbf{u}^1 e^{2t} + 5c_2 \mathbf{u}^2 e^{5t} \quad ,$$

both as $t \to -\infty$ and as $t \to +\infty$. Remember, this vector is tangent to the trajectory at $(x(t), y(t))$ and points in the direction of travel along the trajectory. The same is true with any positive multiple of $\mathbf{x}'(t)$. In particular, let us be clever and use

$$e^{-2t} \mathbf{x}'(t) = e^{-2t} \left[2c_1 \mathbf{u}^1 e^{2t} + 5c_2 \mathbf{u}^2 e^{5t} \right] = 2c_1 \mathbf{u}^1 + 5c_2 \mathbf{u}^2 e^{3t}$$

and

$$e^{-5t} \mathbf{x}'(t) = e^{-5t} \left[2c_1 \mathbf{u}^1 e^{2t} + 5c_2 \mathbf{u}^2 e^{5t} \right] = 2c_1 \mathbf{u}^1 e^{-3t} + 5c_2 \mathbf{u}^2 \quad .$$

Now let $t \to -\infty$. As already noted, we'll then have $(x(t), y(t)) \to (0, 0)$. In addition,

$$\lim_{t \to -\infty} e^{-2t} \mathbf{x}'(t) = \lim_{t \to -\infty} \left[2c_1 \mathbf{u}^1 + 5c_2 \mathbf{u}^2 e^{3t} \right] = 2c_1 \mathbf{u}^1 = 2c_1 \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

This tells us that, as $t \to -\infty$, the tangent to the trajectory at $(x(t), y(t))$ becomes parallel to the vector $\mathbf{u}^1 = [4, 1]^{\mathsf{T}}$, which, itself, is parallel to the straight half-line trajectory corresponding to $(r_1, \mathbf{u}^1)$. With a little thought and some sketching on your own, you will

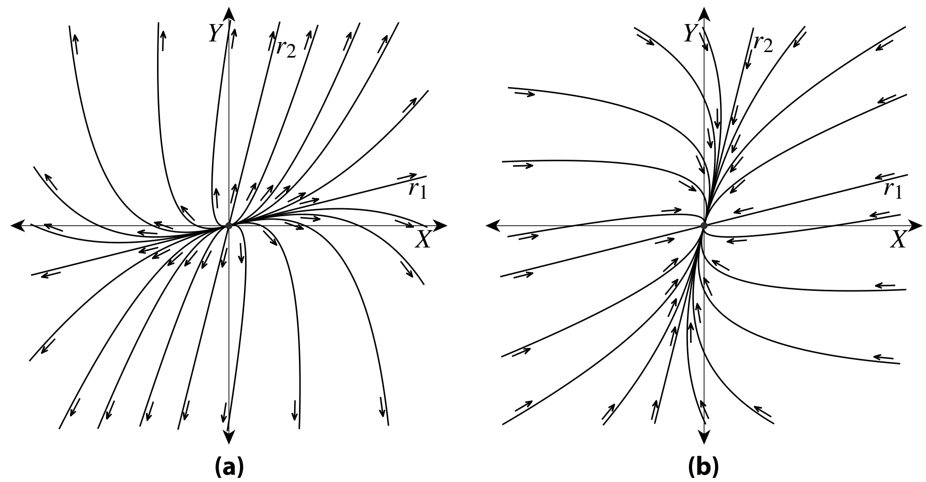


Figure 39.4: Phase portraits for the systems **(a)** in example 39.9 (where $0 < r_1 < r_2$), and **(b)** in example 39.10 (where $r_1 < r_2 < 0$). In each, r_1 and r_2 identify the straight line trajectories corresponding to eigenvalues r_1 and r_2.

realize that this means $(x(t), y(t))$ approaches the origin along a path tangent to the half-line trajectory corresponding to $(r_1, \mathbf{u}^1)$.

Now let $t \to +\infty$. As already noted, we'll then have $(x(t), y(t))$ tracing a path further and further away from the origin. In addition,

$$\lim_{t\to+\infty} e^{-5t}\mathbf{x}'(t) = \lim_{t\to+\infty} \left[2c_1\mathbf{u}^1 e^{-3t} + 5c_2\mathbf{u}^2\right] = 5c_2\mathbf{u}^2 = 5c_2\begin{bmatrix}1\\4\end{bmatrix} .$$

This tells us that, as $t \to +\infty$, the tangent to the trajectory at $(x(t), y(t))$ becomes parallel to the vector $\mathbf{u}^2 = [1, 4]^\mathsf{T}$, which, itself, is parallel to the straight half-line trajectory corresponding to $(r_2, \mathbf{u}^2)$. Thus, as t gets larger, the path being traced out by $(x(t), y(t))$ approaches a path parallel to the straight half-line trajectory corresponding to $(r_2, \mathbf{u}^2)$. However, the trajectory of $\mathbf{x}(t)$ will not get close to this half-line trajectory since the other term, $c_1\mathbf{u}^1 e^{2t}$ is also getting larger, not smaller, as t increases.

Some of these trajectories have been sketched in figure 39.4a.

In general, if $0 < r_1 < r_2$, then a phase portrait for $\mathbf{x}' = \mathbf{A}\mathbf{x}$ can be roughly sketched by doing the following:

1. Plot the critical point $(0, 0)$, and sketch the straight-line trajectories corresponding to $(r_1, \mathbf{u}^1)$ and $(r_2, \mathbf{u}^2)$.

2. Then sketch a reasonable collection of other trajectories with each starting at the origin and tangent there to a straight-line trajectory corresponding to r_1 (the smaller eigenvalue), and then curving around so that it becomes "somewhat parallel" to the nearest straight-line trajectory corresponding to r_2 (the larger eigenvalue). (A minimal direction field may help in refining your sketching of the trajectories.)

3. Be sure to sketch small arrows indicating that the direction of travel along each trajectory is away from the origin.

Again, “node” is the classical term for the critical point $(0, 0)$ in this case, and it is clearly the trajectory of an unstable equilibrium solution. This time (for reasons to given described later) this node is sometimes considered to be “improper”.

By the way, instead of memorizing which end of the trajectories become parallel to which eigenvectors, you may just want to look at which term in

$$\mathbf{x}'(t) \;=\; 2c_1\mathbf{u}^1 e^{2t} \;+\; 5c_2\mathbf{u}^2 e^{5t}$$

becomes dominant as $t \to +\infty$ and $t \to -\infty$ using the fact that

$$e^{\alpha} \;\ll\; e^{\beta} \qquad \text{when} \quad \alpha \ll \beta \quad .$$

We’ll illustrate this approach in the example for the next case.

Trajectories When $r_1 < r_2 < 0$

The analysis here is very similar to that done when $0 < r_1 < r_2$. The main difference is that, because of the negative eigenvalues, the direction of travel along each trajectory will be towards the origin, instead of away. You should do this analysis yourself. You will find that, in general the trajectories generated by

$$\mathbf{x}(t) \;=\; c_1\mathbf{u}^1 e^{r_1 t} \;+\; c_2\mathbf{u}^2 e^{r_2 t} \qquad \text{when} \quad r_1 \;<\; r_2 \;<\; 0$$

can be roughly sketched by doing the following:

1. Plot the critical point $(0, 0)$, and sketch the straight-line trajectories corresponding to $(r_1, \mathbf{u}^1)$ and $(r_2, \mathbf{u}^2)$.
2. Then sketch a reasonable collection of other trajectories with each trajectory starting at the origin and tangent there to a straight-line trajectory corresponding to r_2 (the larger eigenvalue), and then curving around so that it becomes “somewhat parallel” to the nearest straight-line trajectory corresponding to r_1 (the smaller eigenvalue). (Again, a minimal direction field may help in refining your sketching of the trajectories.)
3. Be sure to sketch small arrows indicating that the direction of travel along each trajectory is towards from the origin.

The critical point is, once again, classified as an “improper node” for our system. This time, however, since r_1 and r_2 are negative, we have

$$\lim_{t\to 0} \mathbf{x}(t) \;=\; \lim_{t\to 0}\left[c_1\mathbf{u}^1 e^{r_1 t} \;+\; c_2\mathbf{u}^2 e^{r_2 t}\right] \;=\; c_1\mathbf{u}^1 \cdot 0 \;+\; c_2\mathbf{u}^2 \cdot 0 \;=\; \mathbf{0} \quad .$$

So the critical point $(0, 0)$ is stable. In fact, it is asymptotically stable.

!►*Example 39.10:* *Let’s consider a* 2×2 *system* $\mathbf{x}' = \mathbf{A}\mathbf{x}$ *whose matrix has eigenpairs*

$$(r_1, \mathbf{u}^1) \;=\; \left(-5\,,\, \begin{bmatrix}4\\1\end{bmatrix}\right) \qquad \textit{and} \qquad (r_2, \mathbf{u}^2) \;=\; \left(-2\,,\, \begin{bmatrix}1\\4\end{bmatrix}\right) \quad .$$

In this case, the general solution is

$$\mathbf{x}(t) \;=\; c_1\mathbf{u}^1 e^{r_1 t} \;+\; c_2\mathbf{u}^2 e^{r_2 t} \;=\; c_1 \begin{bmatrix}4\\1\end{bmatrix} e^{-5t} \;+\; c_2 \begin{bmatrix}1\\4\end{bmatrix} e^{-2t} \quad , \tag{39.5}$$

and its derivative is

$$\mathbf{x}'(t) = -5c_1\mathbf{u}^1 e^{-5t} - 2c_2\mathbf{u}^2 e^{-2t} \quad .$$

Again, we start our sketch by plotting a dot at the origin for the equilibrium solution, and sketching the straight-line trajectories corresponding to the eigenpairs $(r_1, \mathbf{u}^1)$ *and* $(r_2, \mathbf{u}^2)$*. Note that the direction of travel on each of these straight-line trajectories is towards the origin. This takes care of the trajectories of solution (39.5) when either* $c_1 = 0$ *or* $c_2 = 0$*.*

Now assume c_1 *and* c_2 *are both nonzero. Because the eigenvalues are both negative, we know (from the above comments) that, as* $t \to +\infty$*, the trajectory for* $\mathbf{x}(t)$ *approaches* $(0, 0)$ *along a path that closely parallels one of the eigenvectors. Rather that try to remember which eigenvector that is, let us just observe that, for large values of* t*,*

$$e^{-5t} \ll e^{-2t} \quad .$$

So the term with e^{-5t} *becomes negligible as* $t \to +\infty$*. A little more explicitly, for large values of* t *we have*

$$\mathbf{x}(t) = c_1\mathbf{u}^1 e^{-5t} + c_2\mathbf{u}^2 e^{-2t} = \left[c_1\mathbf{u}^1 \underbrace{e^{-3t}}_{\approx 0} + c_2\mathbf{u}^2\right]e^{-2t} \approx c_2\mathbf{u}^2 e^{-2t} \quad ,$$

and, by the same approximation,

$$\mathbf{x}'(t) = -5c_1\mathbf{u}^1 e^{-5t} - 2c_2\mathbf{u}^2 e^{-2t} \approx -2c_2\mathbf{u}^2 e^{-2t} = c_2\frac{d}{dt}\left[\mathbf{u}^2 e^{-2t}\right]$$

Clearly, then, as $t \to +\infty$*,* $\mathbf{x}(t)$ *approaches* $(0, 0)$ *along a trajectory that becomes more and more like a straight-line trajectory of a solution corresponding to eigenvalue* $r_2 = -2$*.*

Conversely, this also tells us that the part of the trajectory of $\mathbf{x}(t)$ *corresponding to large negative values of* t *must be far from* $(0, 0)$*, becoming more and more parallel to the the other eigenvector,* $\mathbf{u}^1$ *as* $t \to -\infty$*.*

Using these facts, you should be able to sketch a phase portrait similar to that in figure 39.4b.

(By the way, a similar analysis can be carried out as $t \to -\infty$ *using the fact that*

$$e^{-2t} \ll e^{-5t} \qquad \text{when} \quad 0 \ll t \quad .$$

But there is a proviso: While the e^{-2t} *term in* $\mathbf{x}(t)$ *is negligible compared to the* e^{-5t} *for large negative values of* t*, it is still a very large term which increases as* $t \to -\infty$*. So, while an analysis similar to the above can justify the claim that the path of* $\mathbf{x}(t)$ *becomes more and more parallel to* $\mathbf{u}^1$ *as* $t \to -\infty$*, it does not show that* $\mathbf{x}(t)$ *is actually getting closer to a straight-line trajectory of an solution corresponding to eigenvalue* r_1*.)*

Trajectories When $r_1 < 0 < r_2$

Again, we start with an example that, essentially, explains everything.

!►*Example 39.11:* *Now consider a* 2×2 *system* $\mathbf{x}' = \mathbf{A}\mathbf{x}$ *whose matrix has eigenpairs*

$$(r_1, \mathbf{u}^1) = \left(-2, \begin{bmatrix}4\\1\end{bmatrix}\right) \qquad \text{and} \qquad (r_2, \mathbf{u}^2) = \left(5, \begin{bmatrix}1\\4\end{bmatrix}\right) \quad .$$

The general solution is then

$$\mathbf{x}(t) = c_1\mathbf{u}^1 e^{r_1 t} + c_2\mathbf{u}^2 e^{r_2 t} = c_1\begin{bmatrix}4\\1\end{bmatrix}e^{-2t} + c_2\begin{bmatrix}1\\4\end{bmatrix}e^{5t} \quad . \tag{39.6}$$

We again begin our sketch by plotting a dot at the origin for the equilibrium solution, and sketching the the straight-line trajectories corresponding to the eigenpairs $(r_1, \mathbf{u}^1)$ *and* $(r_2, \mathbf{u}^2)$ *. Note that the direction of travel on the straight-line trajectory corresponding to* $(r_1, \mathbf{u}^1)$ *is towards the origin, while the direction of travel on the straight-line trajectory corresponding to* $(r_2, \mathbf{u}^2)$ *is away from the origin. This takes care of the trajectories of solution (39.6) when either* $c_1 = 0$ *or* $c_2 = 0$ *.*

To sketch any other trajectory, assume c_1 *and* c_2 *are both nonzero, and consider solution (39.6) both when* t *is a large negative value, and when* t *is a large positive value.*

If t *is a large negative value, then* e^{-2t} *is a large positive value while* $e^{5t} \approx 0$ *. So, for large negative values of* t *,*

$$\mathbf{x}(t) = c_1\begin{bmatrix}4\\1\end{bmatrix}e^{-2t} + c_2\begin{bmatrix}1\\4\end{bmatrix}e^{5t} \approx c_1\begin{bmatrix}4\\1\end{bmatrix}e^{-2t} \quad .$$

In other words, that part of the trajectory corresponding to t *being a large negative value is close to one of the straight-line trajectories corresponding to* $r = -2$ *.*

As t *increases, the term with* e^{-2t} *decreases in magnitude while the term with* e^{5t} *increases in magnitude. Once* t *becomes sufficiently large,* e^{-2t} *becomes negligible and,*

$$\mathbf{x}(t) = c_1\begin{bmatrix}4\\1\end{bmatrix}e^{-2t} + c_2\begin{bmatrix}1\\4\end{bmatrix}e^{5t} \approx c_2\begin{bmatrix}1\\4\end{bmatrix}e^{5t} \quad .$$

Hence, this part of the trajectory becomes very close to one of the straight-line trajectories corresponding to $r = 5$ *.*

So, to sketch a trajectory other than that for the equilibrium solution or for one of the straight-line trajectories corresponding to an eigenpair, start your curve close to and nearly parallel to one of the straight line trajectories corresponding to $r = -2$ *and draw it moving towards the origin and curving away from that one straight-line trajectory. As you continue sketching, bend the curve towards a straight-line trajectory for the positive eigenvalue (without crossing any other trajectories) and try to end your curve close to and nearly parallel to that straight-line trajectory. The resulting curve will look a little like a hyperbola with the straight-line trajectories as asymptotes. Be sure to add a little arrow or two to indicate the direction of travel.*

A phase portrait for this system has been sketched in figure 39.5.

In general, if $r_1 < 0 < r_2$, then a phase portrait for $\mathbf{x}' = \mathbf{A}\mathbf{x}$ can be roughly sketched by doing the following:

1. Plot the critical point $(0, 0)$, and sketch the straight-line trajectories corresponding to $(r_1, \mathbf{u}^1)$ and $(r_2, \mathbf{u}^2)$. Note that the direction of travel on those corresponding to the negative eigenvalue is towards the origin, while the direction of travel on those trajectories corresponding to the positive eigenvalue is away from the origin.

2. Then sketch a reasonable collection of other trajectories with each being a roughly hyperbolic shaped curve with the straight-line trajectories as asymptotes.

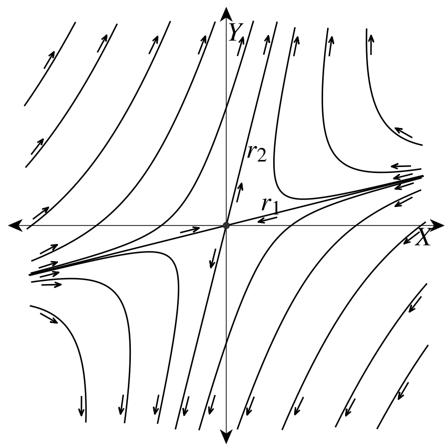


Figure 39.5: A phase portrait of the system in example 39.11 (where $r_1 < 0 < r_2$). Again, r_1 and r_2 identify the straight line trajectories corresponding to eigenvalues r_1 and r_2 .

3. Be sure to sketch small arrows indicating that the direction of travel along each trajectory, with these directions of travel closely matching the directions of travel on the nearby straight-line trajectories.

Finally, at long last, we have an example in which the critical point is not called a node. We call it a *saddle point*. Also note that, while two trajectories do approach $(0,0)$ as $t \to +\infty$, the rest go away from $(0,0)$ as $t \to +\infty$. So, the equilibrium solution for this system is unstable.

Additional Exercises

39.1. *Consider the system*

$$\begin{aligned} x' &= 4x + 2y \\ y' &= 3x - y \end{aligned} \quad .$$

a. *If we write this system in matrix/vector form* $\mathbf{x}' = \mathbf{A}\mathbf{x}$ *, what is the matrix* $\mathbf{A}$ *?*

b. *Verify that the following two vectors are eigenvectors for the matrix* $\mathbf{A}$ *found in the above exercise, and find their corresponding eigenvalues* r_1 *and* r_2 *, respectively.*

$$\mathbf{u}^1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix} \qquad \text{and} \qquad \mathbf{u}^2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad .$$

c. *Using the eigenpairs just found above, write out a pair of solutions to the above system of differential equations, and verify that the Wronskian of this pair is nonzero at* $t = 0$.

d. *Write out a general solution to the above system of differential equations.*

39.2. *Consider the system*

$$\begin{aligned} x' &= 4x - 3y \\ y' &= 6x - 7y \end{aligned} \quad .$$

a. *If we write this system in matrix/vector form* $\mathbf{x}' = \mathbf{A}\mathbf{x}$ *, what is the matrix* $\mathbf{A}$ *?*

b. *Verify that the following two vectors are eigenvectors for the matrix* $\mathbf{A}$ *found in the above exercise, and find their corresponding eigenvalues* r_1 *and* r_2 *, respectively.*

$$\mathbf{u}^1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \quad \text{and} \quad \mathbf{u}^2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \quad .$$

c. *Using the eigenpairs just found above, write out a pair of solutions to the above system of differential equations, and verify that the Wronskian of this pair is nonzero at* $t = 0$ *.*

d. *Write out a general solution to the above system of differential equations.*

39.3. *Consider the system*

$$\begin{aligned} x' &= x - 3y + 3z \\ y' &= 3x - 5y + 3z \\ z' &= 6x - 6y + 4z \end{aligned} \quad .$$

a. *If we write this system in matrix/vector form* $\mathbf{x}' = \mathbf{A}\mathbf{x}$ *, what is the matrix* $\mathbf{A}$ *?*

b. *Verify that the following three vectors are eigenvectors for the matrix* $\mathbf{A}$ *found in the last exercise, and find their corresponding eigenvalues* r_1 *,* r_2 *and* r_3 *, respectively.*

$$\mathbf{u}^1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad , \quad \mathbf{u}^2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \quad \text{and} \quad \mathbf{u}^3 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \quad .$$

c. *Using the eigenpairs found in the above exercises, write out a set of three solutions to the above system of differential equations, and verify that the Wronskian of this pair is nonzero at* $t = 0$ *.*

d. *Write out a general solution to the above system of differential equations.*

39.4. *For each of the following matrices, do the following:*

i. *Find the eigenvalues and a corresponding linearly independent set of eigenvectors for the matrix in the system.*

ii. *List the corresponding eigenpairs.*

iii. *State whether the matrix has a complete set of eigenvectors.*

(Note: Some of the eigenvalues and eigenvectors may be complex.)

a. $\begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix}$ **b.** $\begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}$ **c.** $\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$

d. $\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$ **e.** $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ **f.** $\begin{bmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{bmatrix}$

g. $\begin{bmatrix} 4 & 0 & 1 \\ 0 & 0 & 2 \\ 0 & 8 & 0 \end{bmatrix}$ **h.** $\begin{bmatrix} 4 & 0 & 1 \\ 0 & 0 & -2 \\ 0 & 8 & 0 \end{bmatrix}$ **i.** $\begin{bmatrix} -2 & -1 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

j. $\begin{bmatrix} 4 & 4 & 4 & 4 \\ 0 & 4 & 4 & 4 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$

39.5. *Find a general solution for each of the following systems:*

a. $\mathbf{x}' = \mathbf{Ax}$ *where* $\mathbf{A}$ *is the matrix in exercise 39.4 a*

b. $\mathbf{x}' = \mathbf{Ax}$ *where* $\mathbf{A}$ *is the matrix in exercise 39.4 b*

c. $\mathbf{x}' = \mathbf{Ax}$ *where* $\mathbf{A}$ *is the matrix in exercise 39.4 e*

d. $\mathbf{x}' = \mathbf{Ax}$ *where* $\mathbf{A}$ *is the matrix in exercise 39.4 f*

e. $\mathbf{x}' = \mathbf{Ax}$ *where* $\mathbf{A}$ *is the matrix in exercise 39.4 i*

f. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 1 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

g. $\begin{aligned} x' &= x + 2y \\ y' &= 5x - 2y \end{aligned}$

h. $\begin{aligned} x' &= 8x + 2y \\ y' &= 4x + y \end{aligned}$

i. $\begin{aligned} x' &= 6x - y + 2z \\ y' &= -4x \\ z' &= 3y \end{aligned}$

j. $\begin{aligned} x' &= 2y - 3z \\ y' &= 2x + 7z \\ z' &= 3x + 7y \end{aligned}$

k. $\begin{aligned} x' &= 2x - y + 3z \\ y' &= 2x + y \\ z' &= 2x - y + 3z \end{aligned}$

l. $\begin{aligned} {x_1}' &= -x_1 + 2x_2 \\ {x_2}' &= -2x_2 + 3x_3 \\ {x_3}' &= -3x_3 + 4x_4 \\ {x_4}' &= -4x_4 \end{aligned}$

39.6. *Solve the initial-value problem*

$$\mathbf{x}' = \mathbf{Ax} \qquad \textit{with} \quad \mathbf{x}(0) = \mathbf{x}^0$$

for each of the following choices of $\mathbf{A}$ *and* $\mathbf{x}^0$ *:*

a. $\mathbf{A}$ *is the matrix in exercise 39.4 b and* $\mathbf{x}^0 = [0, -6]^{\mathsf{T}}$

b. $\mathbf{A}$ *is the matrix in exercise 39.4 b and* $\mathbf{x}^0 = [10, 5]^{\mathsf{T}}$

c. $\mathbf{A}$ *is the matrix in exercise 39.4 e and* $\mathbf{x}^0 = [4, -4, 6]^{\mathsf{T}}$

d. $\mathbf{A} = \begin{bmatrix} -2 & 4 \\ 5 & -3 \end{bmatrix}$ *and* $\mathbf{x}^0 = [5, -4]^{\mathsf{T}}$

39.7. *For each of the following, consider the* 2×2 *constant matrix system* $\mathbf{x}' = \mathbf{Ax}$ *where* $\mathbf{A}$ *has the given two eigenpairs. Using just the eigenpairs:*

i. *Write out the general solution to the system.*

ii. *Describe the stability of the node at the origin.*

iii. *Sketch a phase portrait.*

a. $\left(2, \begin{bmatrix}1\\0\end{bmatrix}\right)$ *and* $\left(2, \begin{bmatrix}0\\1\end{bmatrix}\right)$

b. $\left(-2, \begin{bmatrix}1\\0\end{bmatrix}\right)$ *and* $\left(-2, \begin{bmatrix}1\\1\end{bmatrix}\right)$

c. $\left(2, \begin{bmatrix}1\\0\end{bmatrix}\right)$ *and* $\left(4, \begin{bmatrix}0\\1\end{bmatrix}\right)$

d. $\left(-2, \begin{bmatrix}1\\0\end{bmatrix}\right)$ *and* $\left(-4, \begin{bmatrix}0\\1\end{bmatrix}\right)$

e. $\left(4, \begin{bmatrix}1\\0\end{bmatrix}\right)$ *and* $\left(2, \begin{bmatrix}0\\1\end{bmatrix}\right)$

f. $\left(3, \begin{bmatrix}4\\1\end{bmatrix}\right)$ *and* $\left(5, \begin{bmatrix}1\\4\end{bmatrix}\right)$

g. $\left(5, \begin{bmatrix}4\\1\end{bmatrix}\right)$ *and* $\left(3, \begin{bmatrix}1\\4\end{bmatrix}\right)$

h. $\left(-3, \begin{bmatrix}4\\1\end{bmatrix}\right)$ *and* $\left(-5, \begin{bmatrix}1\\4\end{bmatrix}\right)$

39.8. *For each of the following, consider the* 2×2 *constant matrix system* $\mathbf{x}' = \mathbf{A}\mathbf{x}$ *where* $\mathbf{A}$ *has the given two eigenpairs, and, using just the eigenpairs:*

i. *Write out the general solution to the system.*

ii. *(Roughly) sketch (by hand) a phase portrait.*

a. $\left(-2, \begin{bmatrix}0\\1\end{bmatrix}\right)$ *and* $\left(2, \begin{bmatrix}1\\0\end{bmatrix}\right)$

b. $\left(-2, \begin{bmatrix}1\\0\end{bmatrix}\right)$ *and* $\left(2, \begin{bmatrix}0\\1\end{bmatrix}\right)$

c. $\left(-3, \begin{bmatrix}-1\\4\end{bmatrix}\right)$ *and* $\left(3, \begin{bmatrix}4\\1\end{bmatrix}\right)$

d. $\left(-3, \begin{bmatrix}4\\1\end{bmatrix}\right)$ *and* $\left(3, \begin{bmatrix}1\\4\end{bmatrix}\right)$

39.9. *For each of the following* 2×2 *systems below:*

i. *Find the general solution.*

ii. *Describe the stability of the equilibrium solution* $\mathbf{x} = 0$, *and state whether the corresponding critical point is a node or saddle point.*

iii. *Sketch a rough phase portrait, using the a minimal direction field to refine your sketch based on the matrix's eigenpairs.*

a. $\begin{bmatrix}x'\\y'\end{bmatrix} = \begin{bmatrix}5 & 1\\3 & 3\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$

b. $\begin{bmatrix}x'\\y'\end{bmatrix} = \begin{bmatrix}6 & 7\\3 & 2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$

c. $\begin{bmatrix}x'\\y'\end{bmatrix} = \begin{bmatrix}4 & 3\\3 & -4\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$

d. $\begin{bmatrix}x'\\y'\end{bmatrix} = \begin{bmatrix}-3 & -3\\3 & -13\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$

e. $\begin{bmatrix}x'\\y'\end{bmatrix} = \begin{bmatrix}5 & -3\\3 & -5\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$

f. $\begin{bmatrix}x'\\y'\end{bmatrix} = \frac{1}{3}\begin{bmatrix}-9 & 2\\9 & -6\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$

g. $\begin{bmatrix}x'\\y'\end{bmatrix} = \frac{1}{3}\begin{bmatrix}6 & 2\\9 & 9\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$

h. $\begin{bmatrix}x'\\y'\end{bmatrix} = \frac{1}{9}\begin{bmatrix}3 & 2\\-3 & 1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$

39.10. *Let* $\mathbf{A}$ *be any constant* 2×2 *matrix with a complete set of eigenvectors, all corresponding to the same eigenvalue* r *. Show that*

$$\mathbf{A} = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} \quad .$$

39.11. *In figures 39.4 and 39.5 many of the trajectories appear to be similar to either parabolas or hyperbolas. In this exercise, we will see that these trajectories are not, in general, actually parabolas or hyperbolas by considering the trajectories of* $\mathbf{x}' = \mathbf{A}\mathbf{x}$ *where*

$$\mathbf{A} = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} \quad .$$

a. *Find the general solution to* $\mathbf{x}' = \mathbf{A}\mathbf{x}$ *where* $\mathbf{A}$ *is as above and* α *and* β *are real numbers.*

b. *Let* $\mathbf{A}$ *be as above with* $0 < \alpha < \beta$ *.*

i. *Sketch a phase portrait for this system.*

ii. *Let* $[x(t), y(t)]^{\mathsf{T}}$ *be a solution to this system which is neither an equilibrium solution nor a solution corresponding to a single eigenpair. Show that, for some constant* c *and every real value* t *,*

$$y(t) = c[x(t)]^p \qquad \text{where} \quad p = \frac{\beta}{\alpha} \quad .$$

iii. *For what choices of* α *and* β *are the trajectories of this system parabolas (excluding the critical point and straight line trajectories)?*

c. *Let* $\mathbf{A}$ *be as above with* $\alpha < 0 < \beta$ *.*

i. *Sketch a phase portrait for this system.*

ii. *Let* $[x(t), y(t)]^{\mathsf{T}}$ *be a solution to this system which is neither an equilibrium solution nor a solution corresponding to a single eigenpair. Show that, for some constant* c *and every real value* t *,*

$$y(t) = c[x(t)]^p \qquad \text{where} \quad p = \frac{\beta}{\alpha} \quad .$$

iii. *For what choices of* α *and* β *are the trajectories of of this system hyperbolas (excluding the critical point and straight line trajectories)?*

39.12. *Let* $\mathbf{A}$ *be a constant* $N \times N$ *matrix. Show that*

$$\det \mathbf{A} = 0 \qquad \Longleftrightarrow \qquad \mathbf{A} \text{ has a zero eigenvalue} \quad .$$

(There are several ways to prove this. A particularly simple approach is to use the characteristic equation $\det[\mathbf{A} - r\mathbf{I}] = 0$ *.)*

39.13. *In the text, we did not discuss trajectories when an eigenvalue is zero. So:*

a. *Suppose* $(0, \mathbf{u})$ *is an eigenpair for an* $N \times N$ *matrix* $\mathbf{A}$ *. Demonstrate that every point on the straight line through the origin and parallel to* $\mathbf{u}$ *is a critical point for* $\mathbf{x}' = \mathbf{A}\mathbf{x}$ *.*

b. *Suppose* $(0, \mathbf{u}^1)$ *and* $(r, \mathbf{u}^2)$ *are eigenpairs for a* 2×2 *matrix* $\mathbf{A}$, *with* r *being a nonzero real number.*

i. *What are the trajectories of the nonequilibrium solutions to* $\mathbf{x}' = \mathbf{A}\mathbf{x}$? *And what is the effect of the sign of* r ?

ii. *What can be said about* $\mathbf{A}$ *and the phase portrait of* $\mathbf{x}' = \mathbf{A}\mathbf{x}$ *if* $r = 0$ *and* $\{\mathbf{u}^1, \mathbf{u}^2\}$ *are linearly independent?*

c. *For each of the following* 2×2 *systems below:*

i. *Find the general solution.*

ii. *Describe the stability of the equilibrium solutions.*

iii. *Sketch a phase portrait, using the appropriate eigenpairs.*

i. $\begin{aligned} x' &= 8x + 2y \\ y' &= 4x + y \end{aligned}$

ii. $\begin{aligned} x' &= -x - y \\ y' &= -x - y \end{aligned}$

iii. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 6 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

iv. $\begin{aligned} x' &= 3x + y \\ y' &= 6x + 2y \end{aligned}$

v. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & a \\ a & a \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ *where* a *is a nonzero real number.*

Some Answers to Some of the Exercises

WARNING! Most of the following answers were prepared hastily and late at night. They have not been properly proofread! Errors are likely!

1a. $\mathbf{A} = \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix}$
1b. $r_1 = -2\ ,\ r_2 = 5$
1d. $\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ -3 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{5t}$
2a. $\mathbf{A} = \begin{bmatrix} 4 & -3 \\ 6 & -7 \end{bmatrix}$
2b. $r_1 = 2\ ,\ r_2 = -5$
2d. $\mathbf{x}(t) = c_1 \begin{bmatrix} 3 \\ 2 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{-5t}$
3a. $\mathbf{A} = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}$
3b. $r_1 = -2\ , r_2 = -2\ , r_3 = 4$
3d. $\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} e^{-2t} + c_3 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} e^{4t}$
4a. Eigenvalues: $3, -3$; Eigenvector set: $\left\{[1,1]^{\mathsf{T}}, [1,-1]^{\mathsf{T}}\right\}$; Eigenpairs: $\left(3, [1,1]^{\mathsf{T}}\right)$ and $\left(-3, [1,-1]^{\mathsf{T}}\right)$; The matrix has a complete set of eigenvectors.
4b. Eigenpairs: $\left(1, [1,-1]^{\mathsf{T}}\right)$ and $\left(4, [2,1]^{\mathsf{T}}\right)$; The matrix has a complete set of eigenvectors.
4c. Eigenpairs: $\left(0, [1,0]^{\mathsf{T}}\right)$; The matrix does not have a complete set of eigenvectors.
4d. Eigenpairs: $\left(2i, [1,i]^{\mathsf{T}}\right)$ and $\left(-2i, [1,-i]^{\mathsf{T}}\right)$; The matrix has a complete set of eigenvectors.
4e. Eigenpairs: $\left(-1, [1,-2,1]^{\mathsf{T}}\right)$, $\left(1, [1,0,-1]^{\mathsf{T}}\right)$ and $(2, [1,1,1])$; The matrix has a complete set of eigenvectors.
4f. Eigenpairs: $\left(0, [1,-1,0]^{\mathsf{T}}\right)$, $\left(0, [1,0,-1]^{\mathsf{T}}\right)$ and $(9, [1,1,1])$; The matrix has a complete set of eigenvectors.
4g. Eigenpairs: $\left(-4, [1,4,-8]^{\mathsf{T}}\right)$ and $\left(4, [1,0,0]^{\mathsf{T}}\right)$; The matrix does not have a complete set of eigenvectors.
4h. Eigenpairs: $\left(4, [1,0,0]^{\mathsf{T}}\right)$, $\left(4i, [-1, 2+2i, 4-4i]^{\mathsf{T}}\right)$ and $\left(-4i, [-1, 2-2i, 4+4i]^{\mathsf{T}}\right)$; The matrix has a complete set of eigenvectors.
4i. Eigenpairs: $\left(-2, [1,0,0,0]^{\mathsf{T}}\right)$, $\left(-1, [-1,1,0,0]^{\mathsf{T}}\right)$, $\left(1, [0,0,1,0]^{\mathsf{T}}\right)$ and $\left(2, [0,0,0,1]^{\mathsf{T}}\right)$; The matrix has a complete set of eigenvectors.
4j. Eigenpairs: $\left(4, [1,0,0,0]^{\mathsf{T}}\right)$ and $\left(4, [0,0,1,-1]^{\mathsf{T}}\right)$; The matrix does not have a complete set of eigenvectors.
5a. $c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-3t}$
5b. $c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{4t}$
5c. $c_1 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} e^{t} + c_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{2t}$
5d. $c_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{9t}$
5e. $c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} e^{-t} + c_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} e^{t} + c_3 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} e^{t}$
5f. $c_1 \begin{bmatrix} -3 \\ 1 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{7t}$
5g. $c_1 \begin{bmatrix} -2 \\ 5 \end{bmatrix} e^{-4t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t}$
5h. $c_1 \begin{bmatrix} 1 \\ -4 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{9t}$
5i. $c_1 \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ -2 \\ -3 \end{bmatrix} e^{2t} + c_3 \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} e^{9t}$
5j. $c_1 \begin{bmatrix} 7 \\ -3 \\ -2 \end{bmatrix} + c_2 \begin{bmatrix} 7 \\ 3-7\sqrt{11} \\ 2 \end{bmatrix} e^{-2\sqrt{11}t} + c_3 \begin{bmatrix} 7 \\ 3+7\sqrt{11} \\ 2 \end{bmatrix} e^{2\sqrt{11}t}$
5k. $c_1 \begin{bmatrix} -3 \\ 9 \\ 5 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} e^{2t} + c_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{4t}$
5l. $c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \end{bmatrix} e^{-2t} + c_3 \begin{bmatrix} 3 \\ -3 \\ 1 \\ 1 \end{bmatrix} e^{-3t} + c_3 \begin{bmatrix} -4 \\ 6 \\ -4 \\ 1 \end{bmatrix} e^{-4t}$

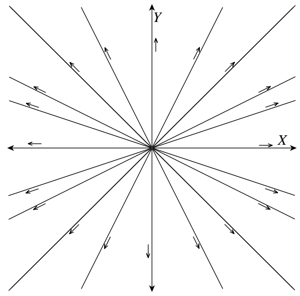
6a. $4\begin{bmatrix}1\\-1\end{bmatrix}e^{t} - 2\begin{bmatrix}2\\1\end{bmatrix}e^{4t}$

6b. $5\begin{bmatrix}2\\1\end{bmatrix}e^{4t}$

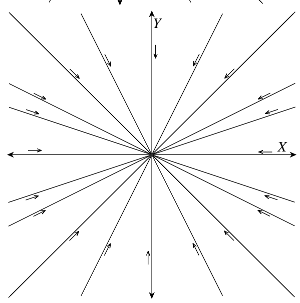
6c. $3\begin{bmatrix}1\\-2\\1\end{bmatrix}e^{-t} - \begin{bmatrix}1\\0\\-1\end{bmatrix}e^{t} + 2\begin{bmatrix}1\\1\\1\end{bmatrix}e^{2t}$

6d. $\begin{bmatrix}4\\-5\end{bmatrix}e^{-7t} + \begin{bmatrix}1\\1\end{bmatrix}e^{2t}$

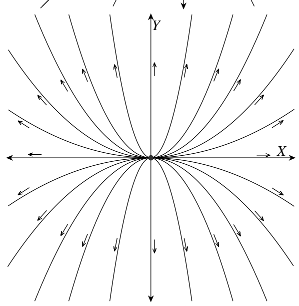
7a. $c_1\begin{bmatrix}1\\0\end{bmatrix}e^{2x} + c_2\begin{bmatrix}0\\1\end{bmatrix}e^{2x}$
Unstable



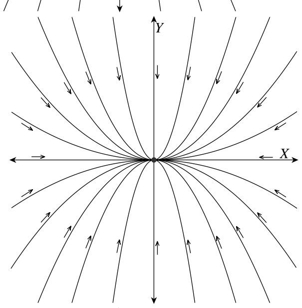
7b. $c_1\begin{bmatrix}1\\0\end{bmatrix}e^{-2x} + c_2\begin{bmatrix}0\\1\end{bmatrix}e^{-2x}$
Asymptotically stable



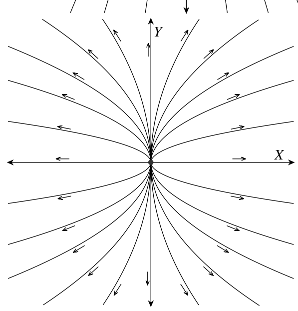
7c. $c_1\begin{bmatrix}1\\0\end{bmatrix}e^{2x} + c_2\begin{bmatrix}0\\1\end{bmatrix}e^{4x}$
Unstable



7d. $c_1\begin{bmatrix}1\\0\end{bmatrix}e^{-2x} + c_2\begin{bmatrix}0\\1\end{bmatrix}e^{-4x}$
Asymptotically stable

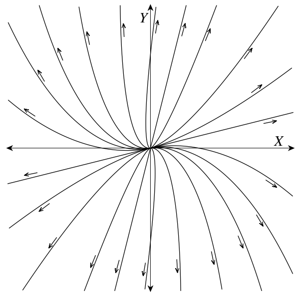


7e. $c_1\begin{bmatrix}1\\0\end{bmatrix}e^{4x} + c_2\begin{bmatrix}0\\1\end{bmatrix}e^{2x}$
Unstable



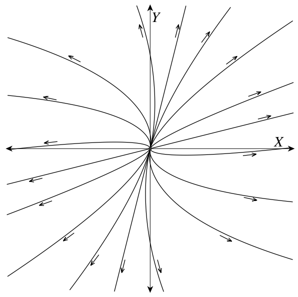
7f. $c_1 \begin{bmatrix} 4 \\ 1 \end{bmatrix} e^{3x} + c_2 \begin{bmatrix} 1 \\ 4 \end{bmatrix} e^{5x}$

Unstable



7g. $c_1 \begin{bmatrix} 4 \\ 1 \end{bmatrix} e^{5x} + c_2 \begin{bmatrix} 1 \\ 4 \end{bmatrix} e^{3x}$

Unstable

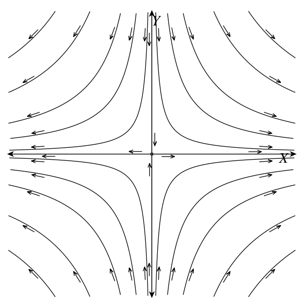


7h. $c_1 \begin{bmatrix} 4 \\ 1 \end{bmatrix} e^{-3x} + c_2 \begin{bmatrix} 1 \\ 4 \end{bmatrix} e^{-5x}$

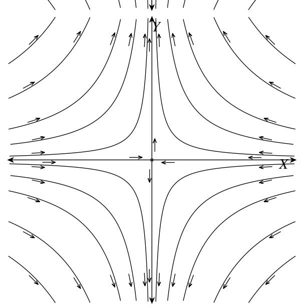
Asymptotically stable

Phase Portrait is that of **f** with the arrows reversed.

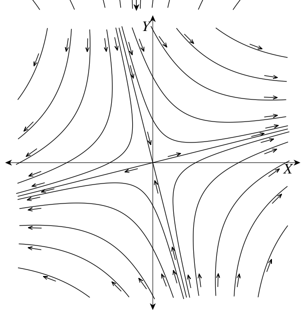
8a. $c_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-2x} + c_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{2x}$



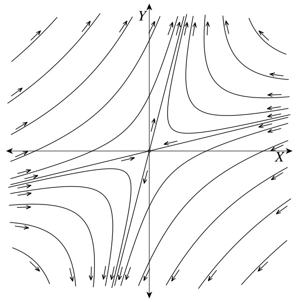
8b. $c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-2x} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{2x}$



8c. $c_1 \begin{bmatrix} -1 \\ 4 \end{bmatrix} e^{-3x} + c_2 \begin{bmatrix} 4 \\ 1 \end{bmatrix} e^{3x}$

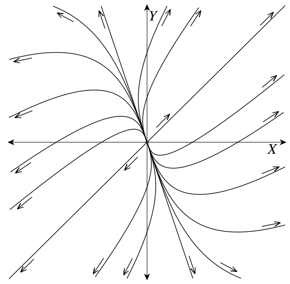


8d. $c_1 \begin{bmatrix} 4 \\ 1 \end{bmatrix} e^{-3x} + c_2 \begin{bmatrix} 1 \\ 4 \end{bmatrix} e^{3x}$



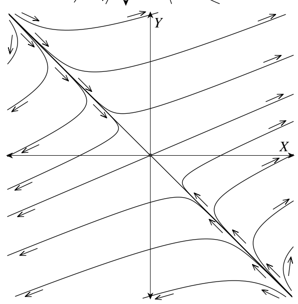
9a. $c_1 \begin{bmatrix} 1 \\ -3 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{6t}$

An unstable node



9b. $c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 7 \\ 3 \end{bmatrix} e^{9t}$

An unstable saddle point



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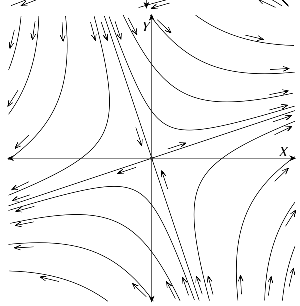
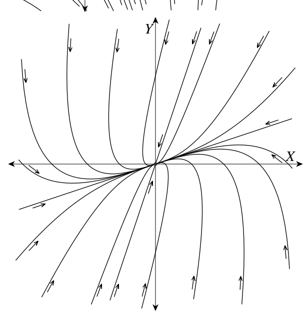
9c. $c_1 \begin{bmatrix} 1 \\ -3 \end{bmatrix} e^{-5t} + c_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix} e^{5t}$

An unstable saddle point



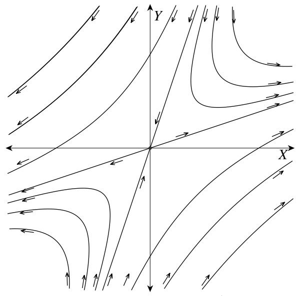
9d. $c_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{-12t} + c_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix} e^{-4t}$

An asymptotically stable node



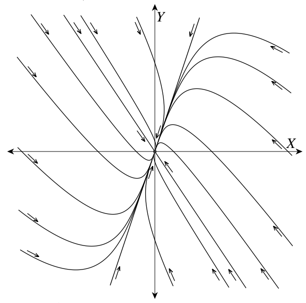
9e. $c_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{-4t} + c_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix} e^{4t}$

An unstable saddle point



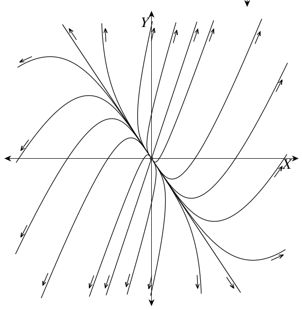
9f. $c_1 \begin{bmatrix} -2 \\ 3 \end{bmatrix} e^{-4t} + c_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{-t}$

An asymptotically stable node



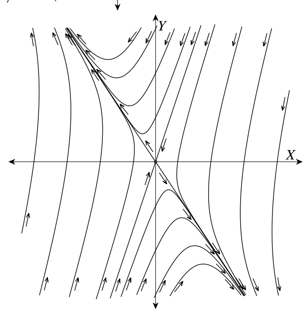
9g. $c_1 \begin{bmatrix} -2 \\ 3 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{4t}$

An unstable node



9h. $c_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{-4t} + c_2 \begin{bmatrix} -2 \\ 3 \end{bmatrix} e^{t}$

An unstable saddle point



11b iii. Only if $\beta = 2\alpha$.

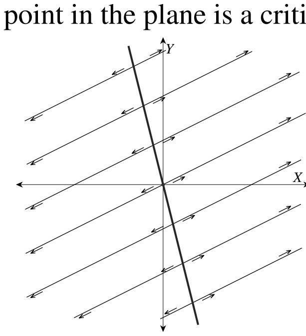
11c iii. Only if $\alpha = -\beta$.

13b i. Each is a straight half line parallel to $\mathbf{u}^2$ with endpoint at a critical point on the line through the origin and parallel to $\mathbf{u}^1$, with the direction of travel towards the critical point if $r < 0$ and away from the critical point if $0 < r$.

13b ii. $\mathbf{A} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and every point in the plane is a critical point.

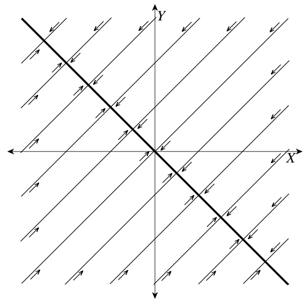
13c i. $c_1 \begin{bmatrix} 1 \\ -4 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{9t}$

Unstable



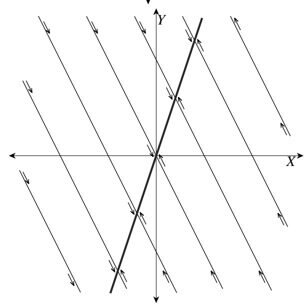
13c ii. $c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t}$

Stable



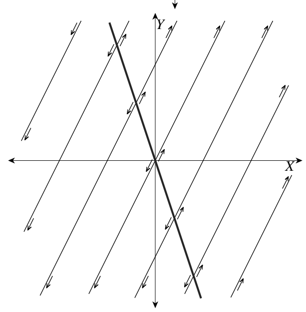
13c iii. $c_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-5t}$

Stable



13c iv. $c_1 \begin{bmatrix} 1 \\ -3 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{5t}$

Unstable



13c v. $c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2at}$

Unstable if $a > 0$; stable if $a < 0$.