37

Standard First-Order Systems: Basics

In the last chapter, we saw that a certain type of first-order system — in which the first derivative of each unknown function is some formula of the collection of unknown functions — naturally arises in a number of applications. We also saw that many other differential equations and systems of differential equations can be rewritten as such first-order systems. Consequently, if we have tools to effectively deal with these first-order systems, then we can also deal with a wide variety of other differential equations and systems by converting them to first-order systems and then using those tools. This suggests narrowing our discussions to those first-order systems, and that's just what we will do in this chapter.

In particular, we will identify more precisely the types of systems we will be studying for the next several chapters, develop more terminology and notation, and discuss the idea of "graphing" solutions of these systems. All of this will be used repeatedly in the following chapters. We might even briefly address questions regarding the existence and uniqueness of solutions. That too is important, at least to reassure ourselves that there are solutions to be found or, at least, somehow analyzed.

37.1 "Standard" First-Order Systems

Basic Terminology

Our main interest will be in $N \times N$ systems of differential equations that can be written as

$$\begin{aligned} {x_1}' &= f_1(t, x_1, x_2, \ldots, x_N) \\ {x_2}' &= f_2(t, x_1, x_2, \ldots, x_N) \\ &\vdots \\ {x_N}' &= f_N(t, x_1, x_2, \ldots, x_N) \end{aligned} \tag{37.1}$$

where N is some positive integer, the x_j's are (presumably unknown) real-valued functions of t (hence $x' = {}^{dx}/_{dt}$), and the $f_k(t, x_1, x_2, \ldots, x_N)$'s — the *component functions* of the system — are known functions of $N + 1$ variables. Note that each equation is a first-order differential equation in which only one of the unknown functions is differentiated. We will refer to such systems of differential equations as *standard first-order systems*.

As in the previous chapter, we will use whatever symbols are convenient for the unknown functions and the component functions. In particular, if $N = 2$ or $N = 3$, we'll usually avoid subscripts and write our generic system as

$$\begin{aligned} x' &= f(t, x, y) \\ y' &= g(t, x, y) \end{aligned} \qquad \text{or} \qquad \begin{aligned} x' &= f(t, x, y, z) \\ y' &= g(t, x, y, z) \\ z' &= h(t, x, y, z) \end{aligned} \tag{37.2}$$

as appropriate.

As already noted, many systems of interest either are or can be converted to standard first-order standard systems. In addition, many of these systems are also "regular", "autonomous" and/or "linear", and just how we deal with a particular system will depend on which of these terms apply. So let us define them:

- A "regular" system is simply a standard system having "reasonably nice" component functions. More precisely, when we refer to a system as being *regular*, we mean that it is a standard first-order system whose component functions and all first partial derivatives of the component functions exist and are continuous for all real values of their variables. Almost all of the systems we will see in the next several chapters will be regular, and by assuming regularity in general, we will be assured (via theorems to be discussed later) that solutions are "reasonably well behaved".

- A standard system is *autonomous* if and only if no component function actually depends on t. When we limit ourselves to autonomous systems, the k^{th} equation in system (37.1) reduces to

  $$x_k{}' = f_k(x_1, x_2, \ldots, x_N) \quad ,$$

  and the systems in set (37.2) reduce to

  $$\begin{aligned} x' &= f(x, y) \\ y' &= g(x, y) \end{aligned} \qquad \text{or} \qquad \begin{aligned} x' &= f(x, y, z) \\ y' &= g(x, y, z) \\ z' &= h(x, y, z) \end{aligned} \quad .$$

  Autonomous systems naturally arise in applications. If you check, all the first-order systems derived in the previous chapter from applications were autonomous.

- A first-order *linear* system is a system that can can be written in the form of system (37.1) with each component function f_k being given by

  $$f_k(t, x_1, x_2, \ldots, x_N) = p_{k1}x_1 + p_{k2}x_2 + \cdots + p_{kN}x_N + g_k$$

  where the p_{kj}'s and g_k's are either constants or functions of t only. If, in addition, all the g_k's are zero, so that our system looks like

  $$\begin{aligned} x_1{}' &= p_{11}x_1 + p_{12}x_2 + \cdots + p_{1N}x_N \\ x_2{}' &= p_{21}x_1 + p_{22}x_2 + \cdots + p_{2N}x_N \\ &\ \vdots \\ x_N{}' &= p_{N1}x_1 + p_{N2}x_2 + \cdots + p_{NN}x_N \end{aligned} \quad , \tag{37.3}$$

then we say our linear system is *homogeneous*. Otherwise, if we have

$$
\begin{aligned}
x_1{}' &= p_{11}x_1 + p_{12}x_2 + \cdots + p_{1N}x_N + g_1 \\
x_2{}' &= p_{21}x_1 + p_{22}x_2 + \cdots + p_{2N}x_N + g_2 \\
&\ \vdots \\
x_N{}' &= p_{N1}x_1 + p_{N2}x_2 + \cdots + p_{NN}x_N + g_N
\end{aligned}
\tag{37.4}
$$

with one or more of the g_k's being nonzero, then we say that the linear system is *nonhomogeneous*.

Observe that, if our system is both autonomous and homogeneous, then the p_{kj}'s in systems (37.3) and (37.4) depend on neither t nor the x_k's, and, thus, are constants. Traditionally, we also refer to such systems as being *linear systems with constant coefficients* or, for reasons soon to be obvious, *constant matrix systems*. It probably will not surprise you to learn that these systems will be the easiest to deal with.

Matrix/Vector Notation for Systems

We can express our generic systems much more concisely if we view the functions in any ordered set of N functions — $x_1(t)$, $x_2(t)$, ... and $x_N(t)$ — as components in the $N \times 1$ matrix

$$
\mathbf{x} = \mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_N(t) \end{bmatrix} .
$$

Following standard conventions and to avoid rather cumbersome terminology later, we will (unless otherwise indicated) use the terms *column vector* or just *vector* as synonyms for "$N \times 1$ matrix".[1] If each component of a given column vector $\mathbf{x}$ is a constant, then we will refer to $\mathbf{x}$ as a *constant vector*. If the components are functions (on some given interval), then we will refer to $\mathbf{x}$ as a *vector-valued function* (on that interval). Typically, we will assume that $\mathbf{x}(t)$ is *componentwise differentiable*; that is, we will assume (usually without comment) that each component of $\mathbf{x}$ is a differentiable function (on the given interval).[2]

Under this viewpoint, we can treat our sets of functions as (column) vector-valued functions with

$$
\mathbf{x}' = \frac{d\mathbf{x}}{dt} = \frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_N(t) \end{bmatrix} = \begin{bmatrix} dx_1/dt \\ dx_2/dt \\ \vdots \\ dx_N/dt \end{bmatrix} = \begin{bmatrix} x_1{}'(t) \\ x_2{}'(t) \\ \vdots \\ x_N{}'(t) \end{bmatrix} .
$$

We can then express system (37.1) as

$$
\mathbf{x}'(t) = \mathbf{F}(t, \mathbf{x}) \qquad \text{or even} \qquad \mathbf{x}' = \mathbf{F} \tag{37.1$'$}
$$

[1] Warning: This definition of a vector as a column matrix is a much more limited definition of "vector" than typically found in, say, texts on linear algebra or physics.

[2] The validity of this assumption of differentiability will be discussed further in section 37.4.

where it is understood that

$$\mathbf{F} = \mathbf{F}(t,\mathbf{x}) = \begin{bmatrix} f_1(t, x_1, x_2, \ldots, x_N) \\ f_2(t, x_1, x_2, \ldots, x_N) \\ \vdots \\ f_N(t, x_1, x_2, \ldots, x_N) \end{bmatrix} \quad .$$

If, in addition, we let

$$\mathbf{P} = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1N} \\ p_{21} & p_{22} & \cdots & p_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ p_{N1} & p_{N2} & \cdots & p_{NN} \end{bmatrix} \quad \text{and} \quad \mathbf{g} = \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_N \end{bmatrix} \quad ,$$

and recall that, using the standard rules of matrix multiplication,

$$\mathbf{Px} = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1N} \\ p_{21} & p_{22} & \cdots & p_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ p_{N1} & p_{N2} & \cdots & p_{NN} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} = \begin{bmatrix} p_{11}x_1 + p_{12}x_2 + \cdots + p_{1N}x_N \\ p_{21}x_1 + p_{22}x_2 + \cdots + p_{2N}x_N \\ \vdots \\ p_{N1}x_1 + p_{N2}x_2 + \cdots + p_{NN}x_N \end{bmatrix} \quad ,$$

then we can rewrite linear systems (37.3) and (37.4), respectively, as

$$\mathbf{x}' = \mathbf{Px} \tag{37.3$'$}$$

and

$$\mathbf{x}' = \mathbf{Px} + \mathbf{g} \quad . \tag{37.4$'$}$$

As with "vectors", we will refer to any matrix $\mathbf{P}$ as being a constant matrix or a matrix-valued function (on some interval) according to whether the components of $\mathbf{P}$ are constants or can be functions (on the given interval), and if all the components of $\mathbf{P}$ are continuous functions on some interval, then we will say $\mathbf{P}$ is a continuous matrix-valued function on that interval. Let us also agree that (unless otherwise indicated) all components of our matrices and vectors are real-valued. If it seems particularly relevant, we'll explicitly say that a given matrix is *real* or *real-valued* to indicate that its components are real values or real-valued functions.

In the future, we will use whatever bold-faced letters or symbols seem convenient to denote vectors and matrices, just as we will use whatever symbols are convenient for the unknown and component functions. In particular, when $N = 2$, we will often use the index-free notation

$$\mathbf{x} = \mathbf{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \quad \text{and} \quad \mathbf{F} = \mathbf{F}(t,\mathbf{x}) = \begin{bmatrix} f(t,x,y) \\ g(t,x,y) \end{bmatrix} \quad .$$

When not using this index-free notation, though, let us at agree that (unless otherwise indicated) whenever a particular bold-faced letter or symbol is used for a particular vector or matrix, then the same letter or symbol — lower-case, nonbold-faced and appropriately subscripted — will denote the components of that vector or matrix. So if we have vector $\mathbf{y}$ and matrix $\mathbf{A}$, then (unless otherwise indicated)

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix} \quad \text{and} \quad \mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N1} & a_{N2} & \cdots & a_{NN} \end{bmatrix} \quad .$$

Let us also agree that, when discussing a standard $N \times N$ system, every vector and matrix under discussion consisting of a single row or column has N components, and every matrix under discussion not consisting of a single row or column is a $N \times N$ matrix.

Along these lines, recall that the *transpose* of a matrix $\mathbf{A}$ — denoted $\mathbf{A}^{\mathsf{T}}$ — is the matrix constructed by interchanging the rows and columns of $\mathbf{A}$. In particular,

$$[x_1\,,\, x_2\,,\, x_3\,,\, \dots\,,\, x_N]^{\mathsf{T}} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_N \end{bmatrix} \quad .$$

For now, the main value in using the transpose is to save a little space by writing column vectors as transposed row matrices.

Some of the advantages of the matrix/vector notation are obvious. It saves a great deal of effort and space, especially when discussing generic systems. Later, when we begin to incorporate elements of linear algebra in nontrivial ways to solve these systems and graph their solutions, the benefits of this notation will become even more significant.

!►*Example 37.1:* *Consider the system*

$$\begin{aligned} x' &= x + 2y \\ y' &= 5x - 2y \end{aligned}$$

along with the initial conditions

$$x(0) = 0 \quad \text{and} \quad y(0) = 1 \quad .$$

In example 36.3 we saw that a solution to this initial-value problem is the pair

$$x(t) = \frac{2}{7}e^{3t} - \frac{2}{7}e^{-4t} \quad \text{and} \quad y(t) = \frac{2}{7}e^{3t} + \frac{5}{7}e^{-4t} \quad .$$

In matrix/vector form, this initial-value problem can be written as either

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x + 2y \\ 5x - 2y \end{bmatrix} \quad \text{with} \quad \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

or as

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 5 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{with} \quad \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

or even as

$$\mathbf{x}' = \mathbf{P}\mathbf{x} \quad \text{with} \quad \mathbf{x}(0) = \mathbf{x}_0$$

where

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} \quad , \quad \mathbf{P} = \begin{bmatrix} 1 & 2 \\ 5 & -2 \end{bmatrix} \quad \text{and} \quad \mathbf{x}_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad .$$

The given solution can then be written as either

$$\mathbf{x}(t) = \begin{bmatrix} \frac{2}{7}e^{3t} - \frac{2}{7}e^{-4t} \\ \frac{2}{7}e^{3t} + \frac{5}{7}e^{-4t} \end{bmatrix} \quad \text{or} \quad \mathbf{x}(t) = \left[\frac{2}{7}e^{3t} - \frac{2}{7}e^{-4t} \quad , \quad \frac{2}{7}e^{3t} + \frac{5}{7}e^{-4t} \right]^{\mathsf{T}}$$

depending on how wasteful of space we care to be. While we are at it, let us also observe that the above formula for $\mathbf{x}$ *can also be written as*

$$\mathbf{x}(t) \;=\; \frac{2}{7}\begin{bmatrix}1\\1\end{bmatrix}e^{3t} \;+\; \frac{1}{7}\begin{bmatrix}2\\5\end{bmatrix}e^{-4t} \quad .$$

(The significance of this last observation will become more apparent in the next few chapters.)

In the future, we will use, or not use, the matrix/vector notation as convenient.

Constant (or Equilibrium) Solutions

A solution

$$\mathbf{x} \;=\; \mathbf{x}(t) \;=\; \left[x_1(t)\,,\; x_2(t)\,,\; \ldots\,,\; x_N(t)\right]^{\mathsf{T}}$$

to a system $\mathbf{x}' = \mathbf{F}(t,\mathbf{x})$ is a *constant solution* if each $x_k(t)$ is simply some single constant value x_k^0 for all t. Now remember,

$$x_k(t) \;=\; \text{some constant} \qquad \text{for all} \quad t \qquad \Longleftrightarrow \qquad {x_k}'(t) \;=\; 0 \qquad \text{for all} \quad t \quad .$$

Thus, $\mathbf{x}$ is a constant solution if and only if

$$\mathbf{x}'(t) \;=\; \mathbf{0} \qquad \text{for all} \quad t$$

where $\mathbf{0}$ is the column vector whose every component is 0. Combining this with $\mathbf{x}' = \mathbf{F}(t,\mathbf{x})$, we get

$$\mathbf{0} \;=\; \mathbf{x}' \;=\; \mathbf{F}(t,\mathbf{x}) \quad .$$

Clearly, then, $\mathbf{x}(t)$ is a constant solution of our system if and only if $\mathbf{x}$ is a constant vector satisfying

$$\mathbf{0} \;=\; \mathbf{F}(t,\mathbf{x}) \qquad \text{for all} \quad t \quad .$$

And from this you find the constant solutions for a given system. (Do remember that we are insisting our solutions be real valued. So the constants must be real numbers.)

Constant solutions will be especially important when we study autonomous systems. And, when the system is autonomous, it is traditional to call any constant solution an *equilibrium solution*. We will follow tradition.

!►*Example 37.2:* *Let's try to find every equilibrium solution for the autonomous system*

$$\begin{aligned} x' &= x(y^2 - 9) \\ y' &= (x-1)(y^2+1) \end{aligned} \quad .$$

The constant/equilibrium solutions are all obtained by setting x' *and* y' *both equal to* 0*, and then solving the resulting algebraic system,*

$$\begin{aligned} 0 &= x(y^2 - 9) \\ 0 &= (x-1)(y^2+1) \end{aligned} \quad . \qquad (37.5)$$

Consider the first equation, first:

$$0 = x(y^2 - 9)$$

$$\hookrightarrow \qquad x = 0 \quad \text{or} \quad y^2 = 9$$

$$\hookrightarrow \qquad x = 0 \quad \text{or} \quad y = 3 \quad \text{or} \quad y = -3 \quad .$$

If $x = 0$ *, then the second equation equation in system (37.5) reduces to*

$$0 = (0-1)(y^2+1) \quad ,$$

which means that $y^2 = -1$ *, and, hence,* $y = \pm i$ *. But these are not real numbers, as required. So we do not have an equilibrium solution*

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} x^0 \\ y^0 \end{bmatrix} \qquad \text{with} \quad x^0 = 0 \quad .$$

On the other hand, if the first equation in system (37.5) is satisfied because $y = 3$ *, then the second equation in that system reduces to*

$$0 = (x-1) \cdot 10 \quad ,$$

telling us that $x = 1$ *. Thus, one equilibrium solution for our system of differential equations is*

$$\mathbf{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \qquad \text{for all} \quad t \quad .$$

Finally, if the first equation in system (37.5) holds because $y = -3$ *, then the second equation in that system becomes*

$$0 = (x-1) \cdot 10 \quad .$$

Hence, again, $x = 1$ *, and the corresponding equilibrium solution is*

$$\mathbf{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \end{bmatrix} \qquad \text{for all} \quad t \quad .$$

In summary, then, our system of differential equations has two equilibrium solutions:

$$\mathbf{x}(t) = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \qquad \text{and} \qquad \mathbf{x}(t) = \begin{bmatrix} 1 \\ -3 \end{bmatrix} \quad .$$

Let us make one final observation regarding constant solutions; namely, that the zero vector, $\mathbf{0}$, is always a constant solution to any linear homogeneous system of differential equations

$$\mathbf{x}' = \mathbf{P}\mathbf{x}$$

simply because, for any $N \times N$ matrix $\mathbf{P}$, $\mathbf{P}\mathbf{0} = \mathbf{0}$.

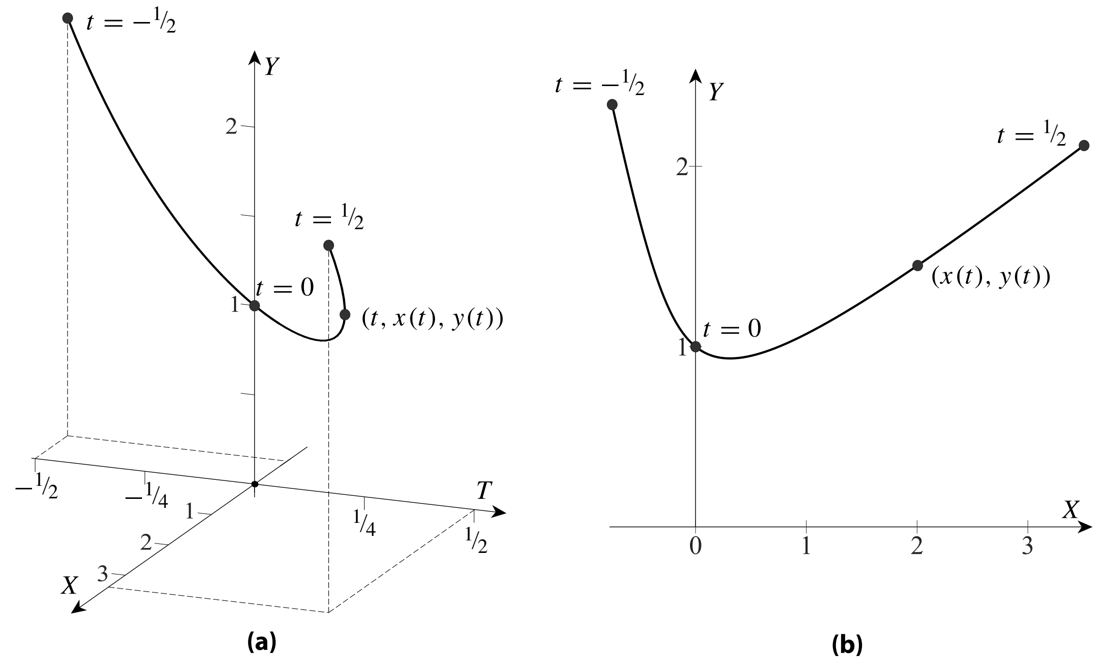

Figure 37.1: Two graphical representations of the solution from example 37.3 with $-1/2 \leq t \leq 1/2$: **(a)** The actual graph, and **(b)** The curve traced out by $(x(t), y(t))$ in the XY–plane.

"Graphing" True Graphs and Trajectories

Let us briefly discuss two ways of graphically representing solutions to standard first-order systems, starting with a simple example.

!►Example 37.3: *Consider "graphically representing" the solution to the initial-value problem*

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x + 2y \\ 5x - 2y \end{bmatrix} \quad \text{with} \quad \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad .$$

From example 36.3 we know a solution to this initial-value problem is the pair

$$x(t) = \frac{2}{7}e^{3t} - \frac{2}{7}e^{-4t} \quad \text{and} \quad y(t) = \frac{2}{7}e^{3t} + \frac{5}{7}e^{-4t} \quad .$$

To construct the actual graph of this solution, we need to plot each point $(t, x(t), y(t))$ in TXY–space, using the above formulas for $x(t)$ and $y(t)$. This results in a curve in TXY–space. Part of this curve has been sketched in figure 37.1a.

As an alternative to constructing the graph, we can sketch the curve in the XY–plane that is traced out by $(x(t), y(t))$ as t varies. That is what what was sketched in figure 37.1b.

Take a look at the two figures. Both were easily done using a computer math package (Maple).

In general, the graph of a solution to an $N \times N$ system requires $N + 1$ axes. Sketching such a graph is do-able if $N = 2$, as in the above example, especially if you have a decent computer math package. Then you can even rotate the image to see the graph from different views. Unfortunately, when you are limited to just one view, as in figure 37.1a, it may be somewhat difficult to properly interpret the figure. Because of this, and because of the very serious problems we would have if $N > 2$, we will rarely, if ever again, actually attempt to sketch true graphs of our solutions.

On the other hand, we will find the approach illustrated by figure 37.1b quite useful, especially for 2×2 systems. Moreover, the sketches that we will generate for 2×2 systems will give us insight as to the behavior of solutions to larger systems.

Observe that the curve in figure 37.1b has a natural "direction of travel" corresponding the way the curve is traced out as t increases. If you start at the point where $t = -1/2$ and travel the curve "in the positive direction" (that is, in the direction in which $(x(t), y(t))$ travels along the curve as t increases), then you would pass through the point where $t = 0$ and then through the point where $t = +1/2$. That makes this an *oriented* curve. We will call this oriented curve a *trajectory* for our system.

Just to be a bit more compleat: Suppose we have a solution

$$\mathbf{x}(t) = \left[x_1(t), x_2(t), \ldots, x_N(t)\right]^{\mathsf{T}}$$

to a standard $N \times N$ system of first-order differential equations $\mathbf{x}' = \mathbf{F}(t, \mathbf{x})$. We will view the components of $\mathbf{x}(t)$, — $x_1(t)$, $x_2(t)$, ... and $x_N(t)$ — as the coordinates of a point in N-dimensional space using a Cartesian coordinate system, and we will refer to the oriented curve traced by out by this point as t increases as this solution's *trajectory*, with the orientation being the direction of travel along the curve given by $(x_1(t), x_2(t), \ldots, x_N(t))$ as t increases. We will also refer to this oriented curve as a *trajectory for the system* $\mathbf{x}' = \mathbf{F}(t, \mathbf{x})$. Much of the analysis in next several chapters will be devoted to determining just what the trajectories of various system look like.

This raises a minor issue of notation and terminology. Traditionally, most people use an N-tuple, such as

$$(x, y) \qquad \text{or} \qquad (x_1, x_2, \ldots, x_N)$$

to describe position, and not a column vector

$$\mathbf{x} = [x, y]^{\mathsf{T}} \qquad \text{or} \qquad \mathbf{x} = [x_1, x_2, \ldots, x_N]^{\mathsf{T}}$$

where the x, y and x_k's are the positions coordinates using some coordinate system. Still, either notation is simply an ordered listing of the coordinates and so, either can be used to describe position. Let us go ahead and do so. This means, for example, that we will accept the phrase

The trajectory traced out by $\mathbf{x}(t)$ *(as* t *varies)*

as meaning

The trajectory traced out by $(x_1(t), y_2(t), \ldots, x_N(t))$ *(as* t *varies)*
where $[x_1(t), x_2(t), \ldots, x_N(t)]^{\mathsf{T}} = \mathbf{x}(t)$

Sometimes, when we want to emphasize the fact that $\mathbf{x}$ is describing position, we will refer to $\mathbf{x}$ as a *position vector* or even as simply a "point (in N-dimensional space)".

"Arrows", Velocity Vectors and the Direction of Travel

Traditionally, a vector $\mathbf{a}$ is often thought of as an "arrow" of a particular length and pointing in a particular direction. We, however, have defined a vector $\mathbf{a}$ to simply be a column matrix

$$\mathbf{a} = \left[a_1, a_2, \ldots, a_N\right]^{\mathsf{T}} \quad .$$

Still, you should be well-acquainted with the ‘arrow’ associated with such a column vector when using a Cartesian coordinate system. In particular, a position vector

$$\mathbf{x} = [x, y]^{\mathsf{T}} \qquad \text{or} \qquad \mathbf{x} = [x_1, x_2, \ldots, x_N]$$

can be viewed, respectively, as an arrow from the origin

$$(0, 0) \qquad \text{or} \qquad (0, 0, \ldots, 0)$$

to the position with coordinates

$$(x, y) \qquad \text{or} \qquad (x_1, x_2, \ldots, x_N) \quad .$$

Do recall that, if $\mathbf{x}$ is a vector-valued function $\mathbf{x}(t)$, then

$$\mathbf{v}(t) = \mathbf{x}'(t) = \left[x_1{}'(t)\,,\; x_2{}'(t)\,,\; \ldots\,,\; x_N{}'(t)\right]^{\mathsf{T}}$$

is the *velocity vector* $\mathbf{v}$ at time t of an object whose position at time t is $\mathbf{x}(t)$. As you should recall from elementary multivariable calculus, this vector $\mathbf{v}(t)$ is an ‘arrow’ pointing in the direction the object is traveling at time t . So, if we pick some value t_0 for t and draw $\mathbf{v}^0 = \mathbf{v}(t_0)$ at position $\mathbf{x}^0 = \mathbf{x}(t_0)$, then $\mathbf{v}^0$ will be tangent to and pointing in the direction of travel of the trajectory of $\mathbf{x}(t)$ at $\mathbf{x}^0$, thus giving some idea of what that trajectory looks like near that point. And if $\mathbf{x}(t)$ is known to be a solution to

$$\mathbf{x}'(t) = \mathbf{F}(t, \mathbf{x}) \quad ,$$

then we can actually compute $\mathbf{v}^0 = \mathbf{x}'(t_0)$ for each choice of t_0 and $\mathbf{x}^0$ without solving the system for $\mathbf{x}(t)$. All we need to do is to compute $\mathbf{F}(t_0, \mathbf{x}^0)$.

!►*Example 37.4:* *Consider the trajectories of two objects both of whose positions at time* t *,* $(x(t), y(t))$ *satisfy the nonautonomous and nonlinear system*

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} t(x + y) \\ y - tx \end{bmatrix} \quad .$$

Assume the first object passes through the point $(x, y) = (1, 2)$ *at time* $t = 0$ *. At that time, its velocity is*

$$\mathbf{v} = \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0(1 + 2) \\ 2 - 0 \cdot 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \quad .$$

However, if the other object also passes through the point $(x, y) = (1, 2)$ *, but at time* $t = 2$ *, then its velocity at that time is*

$$\mathbf{v} = \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2(1 + 2) \\ 2 - 2 \cdot 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \end{bmatrix} \quad .$$

Note the directions of travel for each of these objects as they pass through the point $(x, y) = (1, 2)$ *: The first is moving parallel to the* Y*–axis, while the second in moving parallel to the* X*–axis.*

As the last example illustrates, the direction of travel for a solution's trajectory through a given point may depend on "when" it passes through the point. However, this is only for *non*autonomous systems. If our first-order system $\mathbf{x}' = \mathbf{F}$ is autonomous, then $\mathbf{F}$ does not depend on t, only on the components of $\mathbf{x}$. Consequently,

$$\mathbf{x}' = \mathbf{F}(\mathbf{x})$$

depends only on position. We will use this fact in the next section.

37.2 Sketching Trajectories for Autonomous Systems

The Two-Dimensional Case

Direction Fields

Suppose we are given a regular 2×2 autonomous first-order system of differential equations

$$\begin{aligned} x' &= f(x, y) \\ y' &= g(x, y) \end{aligned} \quad .$$

Now, pick a point (x_0, y_0) on the XY–plane, and, using the given system, compute the 'velocity' at that point for the trajectory through that point,

$$\mathbf{v} = \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} f(x_0, y_0) \\ g(x_0, x_0) \end{bmatrix} \quad .$$

This gives us a vector tangent at the point (x_0, x_0) to any trajectory passing through this point, and pointing in the direction of travel along the trajectory as t increases. So, if we draw a short arrow at (x_0, y_0) in the same direction as this velocity vector, then we have a short arrow tangent to the trajectory through this point and pointing in the "direction of travel" along this curve. We will call this short arrow a *direction arrow*. (This assumes $\mathbf{x}'$ is nonzero at the point. If it is zero, we have a "critical point". We'll discuss critical points in just a little bit.) In figure 37.2a, we've sketched a few of these direction arrows at points along one particular trajectory.

If we sketch a corresponding direction arrow at every point (x_j, y_k) in a grid of points, then we have a *direction field*, as illustrated (along with a few trajectories) in figure 37.2b. This direction field tells us the general behavior of the system's trajectories. To sketch a trajectory using a direction field, simply start at any chosen point in the plane, and sketch a curve following the directions indicated by the nearby direction arrows. "Go with the flow", do not attempt to "connect the dots". The goal is to draw, as well as practical, a curve whose tangent at each point on the curve (other than endpoints) is lined up with the direction arrow that would be sketched at that point.

The construction and use of a direction field for a 2×2 first-order autonomous system of differential equations is analogous to the construction and use of a slope field for a first-order differential equation (see chapter 8). It's not exactly the same — we are now sketching trajectories instead of true graphs of solutions (which would require a T–axis), but the mechanics are very much the same. And, just as with slope lines for a slope fields, it is good for your understanding to practice sketching a few direction arrows, and, for the sake of your sanity, it is a good idea

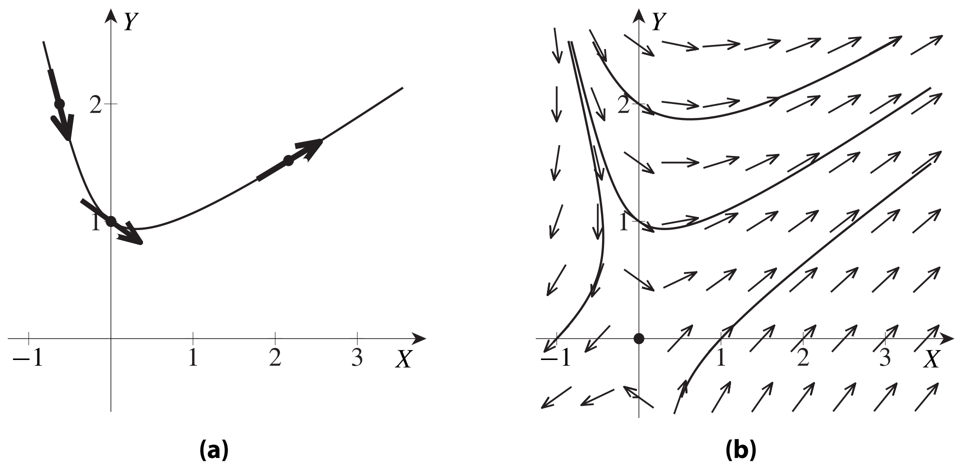

Figure 37.2: Direction arrows for trajectories for the system in example 37.5, with **(a)** being a few direction arrows tangent to the trajectory passing through $(0, 1)$ (drawn oversized for clarity), and **(b)** being a more complete direction field, along with a few more trajectories.

to learn how to construct direction fields (and trajectories) using a good computer math package such as Maple or Mathematica.[3]

Critical Points

When constructing a direction field, it is important to note each point (x_0, y_0) in $\mathbb{R}^2$ for which

$$\begin{bmatrix} f(x_0, y_0) \\ g(x_0, y_0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad .$$

Any such point (x_0, y_0) is said to be a *critical point* for the system. Since the zero vector has no well-defined direction, we cannot sketch a direction arrow at a critical point. Instead, plot a clearly visible dot at this point. After all, if (x_0, y_0) is a critical point for our system, then

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} f(x_0, y_0) \\ g(x_0, y_0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad ,$$

which, in turn, means we have the constant/equilibrium solution

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \qquad \text{for all} \quad t \quad ,$$

and, if you think about it a moment, you'll realize that the point (x_0, y_0) is the entire trajectory of this solution.

In fact, this gives us an alternate definition for critical point; namely, that a *critical point* for an autonomous system of differential equations is the trajectory of an equilibrium solution for that system.

[3] You can also find online programs for constructing direction fields. At the time this was written, a good program, pplane by John Polking, could be found at http:\\ math.rice.edu/∼dfield/dfpp.html.

Finding critical points and determining the behavior of the trajectories in regions around them will prove rather important. The mechanics of finding critical points is identical to the mechanics of finding equilibrium solutions (as illustrated in example 37.2 on page 37–6). Issues regarding the behavior of near-by trajectories will be discussed in the next section, and in future chapters.

!►*Example 37.5:* *Consider, once again, the system*

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x + 2y \\ 5x - 2y \end{bmatrix} \quad .$$

Plugging in $(x, y) = (0, 1)$ *, we get*

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 + 2 \cdot 1 \\ 5 \cdot 0 - 2 \cdot 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix} \quad .$$

Thus, the direction arrow sketched at position $(x, y) = (0, 1)$ *should be a short arrow (which we center at the point) parallel to and pointing in the same direction as the vector from the origin* $(0, 0)$ *to position* $(2, -2)$ *. That is what was sketched at point* $(0, 1)$ *in figure 37.2a, along with the trajectory through that point.*

A more complete direction field, along with three other trajectories, is illustrated in figure 37.2b. It was drawn by Maple (and touched up in a graphics program). Note the dot at $(0, 0)$ *. This is a critical point for the system, and is the trajectory of the one equilibrium solution*

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \textit{for all} \quad t \quad .$$

Trajectories and Solutions

Keep in mind that a trajectory through a point (a, b) for a regular autonomous system

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} f(x, y) \\ g(x, y) \end{bmatrix}$$

is not a solution to the system, it is the path traced out by the solution to the initial-value problem

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} f(x, y) \\ g(x, y) \end{bmatrix} \qquad \text{with} \qquad \begin{bmatrix} x(t_0) \\ y(t_0) \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

for any choice of t_0 . And since there are infinitely many possible values of t_0 , we should expect infinitely many solutions tracing out that one trajectory.

However, all the different solutions corresponding to a single trajectory are simply 'shifted' versions of each other. To see this, let $\left[x^0(t), y^0(t)\right]^{\mathsf{T}}$ be the solution to

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} f(x, y) \\ g(x, y) \end{bmatrix} \qquad \text{with} \qquad \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} \quad .$$

You can then easily verify for yourself that, for any real value t_0, the vector-valued function

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} x^0(t-t_0) \\ y^0(t-t_0) \end{bmatrix}$$

satisfies

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix} \quad \text{with} \quad \begin{bmatrix} x(t_0) \\ y(t_0) \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} \quad .$$

Uniqueness theorems that we will briefly discuss in section 37.4 will then assure us that there are no other solutions to this initial-value problem.

It should also be noted that the 'existence and uniqueness of solutions' will imply a corresponding 'existence and uniqueness of trajectories', allowing us to say that "through each point there is one and only one trajectory". We'll say more about this in section 37.4.

Phase Portraits and Planes

A little more terminology: When dealing with direction fields and trajectories for a standard 2×2 autonomous system, we refer to the plane on which we sketch the direction field and/or trajectories as the *phase plane* (as opposed to the "XY–plane" or "X_1X_2–plane" plane or ...). If we sketch an 'enlightening' representative sample of trajectories on the phase plane, then this sketch is said to be a *phase portrait* of the system. At this point, we are using direction fields to sketch phase portraits, so we are getting phase portraits superimposed on direction fields. If a phase portrait does not have an accompanying direction field to indicate direction of travel along the trajectories, then you should have little arrows on the trajectories to indicate the direction of travel for each trajectory.

Higher-Order Cases

The fundamental ideas just discussed regarding direction fields and trajectories for a 2×2 autonomous system extend naturally to analogous ideas for any $N \times N$ regular autonomous system of first-order differential equations

$$\begin{aligned} {x_1}' &= f_1(x_1, x_2, \dots, x_N) \\ {x_2}' &= f_2(x_1, x_2, \dots, x_N) \\ &\ \vdots \\ {x_N}' &= f_N(x_1, x_2, \dots, x_N) \end{aligned} \quad .$$

As before, we define a *critical point* to be any point $\left(x_1^0, x_2^0, \dots, x_N^0\right)$ in N-dimensional space for which

$$\begin{aligned} f_1\left(x_1^0, x_2^0, \dots, x_N^0\right) &= 0 \\ f_2\left(x_1^0, x_2^0, \dots, x_N^0\right) &= 0 \\ &\ \vdots \\ f_N\left(x_1^0, x_2^0, \dots, x_N^0\right) &= 0 \end{aligned} \quad ,$$

and, as before, any such point is the trajectory of the corresponding equilibrium solution

$$\mathbf{x}(t) = \left[x_1^0 , x_2^0 , \ldots , x_N^0 \right]^{\mathsf{T}} \qquad \text{for all} \quad t$$

for our system $\mathbf{x}' = \mathbf{F}$.

Likewise, at any point other than a critical point, we can, in theory, find a short vector pointing in the direction of travel of any trajectory through that point by just taking any short vector pointing in the same direction as $\mathbf{F}$ computed at that point. This gives a *direction arrow* at that point. Plotting these direction arrows on a suitable grid of points in N-dimensional space then gives us a *direction field* for the system.

Admittedly, few of us can actually sketch and use a direction field when $N > 2$ (especially if $N > 3$!). Still, we can find the critical points, and we will later discover that much of what we learn about the behavior of trajectories near critical points for 2×2 systems will give us insight as to the behavior of trajectories near critical points for many larger systems.

By the way, it is traditional to refer to the N-dimensional space in which the trajectories would, in theory, be drawn as the *phase space* (*phase plane* if $N = 2$), and any enlightening representative sample of trajectories in this space is called a *phase portrait*. Of course, visualizing a phase portrait when $N > 2$ requires a certain imagination (especially if $N > 3$!).

37.3 Critical Points, Stability and Long-Term Behavior

A useful feature of a direction field for an autonomous system of differential equations is that it can give us some notion of the long-term behavior of the solutions to that system. All we need to do is to follow sketched trajectories.

Critical Points and Stability

Stability of Equilibrium Solutions

Of particular interest will be the long-term behavior of solutions whose trajectories pass close to a critical point $\left(x_1^0, x_2^0, \ldots, x_N^0\right)$ of the system, and we will use this behavior to classify the 'stability' of that critical point and the corresponding equilibrium solution

$$\mathbf{x}_{\text{eq}}(t) = \mathbf{x}^0 \qquad \text{for all} \quad t$$

where $\mathbf{x}^0 = \left[x_1^0, x_2^0, \ldots, x_N^0\right]^{\mathsf{T}}$. Loosely speaking we will classify this critical point and the corresponding equilibrium solution as being:

- *stable* if and only if each trajectory that gets close to $\left(x_1^0, x_2^0, \ldots, x_N^0\right)$ stays close to $\left(x_1^0, x_2^0, \ldots, x_N^0\right)$ afterwards. That is, this critical point and equilibrium solution are stable if and only if, whenever $\mathbf{x}$ is any other solution to the system satisfying $\mathbf{x}(t_0) \approx \mathbf{x}^0$ for some t_0, then

  $$\mathbf{x}(t) \approx \mathbf{x}^0 \qquad \text{for all} \quad t > t_0 \quad ,$$

  as illustrated in figures 37.3a and 37.3b.

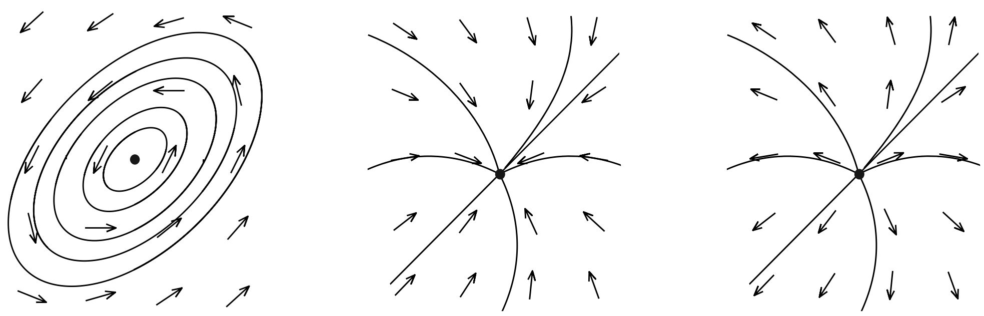

Figure 37.3: Two-dimensional direction fields and trajectories about critical points corresponding to equilibrium solutions that are **(a)** stable, but not asymptotically stable, **(b)** asymptotically stable and **(c)** unstable.

- *asymptotically stable* if and only if each trajectory that gets close to $\left(x_1^0, x_2^0, \ldots, x_N^0\right)$ doesn't just stay close, but converges to $\left(x_1^0, x_2^0, \ldots, x_N^0\right)$ as $t \to +\infty$. That is, this critical point and equilibrium solution are asymptotically stable if and only if, whenever $\mathbf{x}$ is any other solution to the system satisfying $\mathbf{x}(t_0) \approx \mathbf{x}^0$ for some t_0, then

$$\lim_{t \to +\infty} \mathbf{x}(t) = \mathbf{x}^0 \quad ,$$

  as illustrated in figure 37.3b. (Note that an asymptotically stable critical point is automatically stable.)

- *unstable* if and only if the equilibrium solution is not stable. Examples of unstable equilibrium solutions are illustrated in figure 37.3c (in which the nearby trajectories all diverge away from the critical point) and in figure 37.2b (in which trajectories approach the critical point $(0, 0)$ and then diverge away).

Be warned that the stability of an equilibrium solution is not always clear from just the direction field. The field may suggest that the nearby trajectories are loops circling the critical point (indicating that the equilibrium solution is stable) when, in fact, the nearby trajectories are either slowing spiralling in towards or out from the critical point (in which case the equilibrium solution is actually either asymptotically stable or simply unstable).

The identification of the stability of equilibrium solutions turns out to be rather important in the practical study of autonomous systems of differential equations, especially when the system is not linear. We'll return to this issue and develop more definitive ways of determining stability in a few chapters.

Precise Definitions

Of course, precise mathematics requires precise definitions. So, to be precise, we classify our equilibrium solution

$$\mathbf{x}_{\text{eq}}(t) = \mathbf{x}^0 \qquad \text{for all} \quad t$$

and the corresponding critical point as being:

- *stable* if and only if, for each $\epsilon > 0$, there is a corresponding $\delta > 0$ such that, if $\mathbf{x}$ is any solution to the system satisfying[4]

$$\left\|\mathbf{x}(t_0) - \mathbf{x}^0\right\| < \delta \qquad \text{for some} \quad t_0 \quad ,$$

then

$$\left\|\mathbf{x}(t_0) - \mathbf{x}^0\right\| < \epsilon \qquad \text{for all} \quad t > t_0 \quad .$$

- *asymptotically stable* if and only if there is a $\delta > 0$ such that, if $\mathbf{x}$ is any solution to the system satisfying

$$\left\|\mathbf{x}(t_0) - \mathbf{x}^0\right\| < \delta \qquad \text{for some} \quad t_0 \quad ,$$

then

$$\lim_{t\to+\infty} \left\|\mathbf{x}(t_0) - \mathbf{x}^0\right\| = 0 \quad .$$

- *unstable* if the equilibrium solution is not stable.

While we are at it, we should give precise meanings to the 'convergence'/'divergence' of a trajectory to/from a critical point corresponding to an equilibrium solution $\mathbf{x}(t) = \mathbf{x}^0$. So assume we have a trajectory and any solution $\mathbf{x}(t)$ to the system that generates that trajectory. We'll say the the trajectory

- *converges* to the critical point if and only if $\lim_{t\to+\infty} \mathbf{x}(t) = \mathbf{x}^0$, and
- *diverges* from the critical point if and only if $\lim_{t\to-\infty} \mathbf{x}(t) = \mathbf{x}^0$.

Long-Term Behavior

A direction field may also give us some idea of the long-term behavior of the solutions to the given system at points far away from the critical points. Of course, this supposes that any patterns that appear to be evident in the direction field actually continue outside the region on which the direction field is drawn. This issue will be further examined later, at least for some linear systems.

!►*Example 37.6:* *Consider the direction field and sample trajectories sketched in figure 37.2b. In particular, look at the trajectory passing through the point $(x, y) = (1, 0)$, and follow it in the direction indicated by the direction field. The last part of this curve seems to be straightening out to a straight line proceeding further into the first quadrant at, very roughly, a 45 degree angle to both the positive X–axis and positive Y–axis. This suggests that, if $\mathbf{x}(t) = [x(t), y(t)]^{\mathsf{T}}$ is any solution to the direction field's system satisfying $\mathbf{x}(t_0) = [1, 0]^{\mathsf{T}}$ for some t_0, then*

$$\lim_{t\to\infty} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \infty \\ \infty \end{bmatrix}$$

with

$$y(t) \approx x(t) \qquad \text{when} \quad t \text{ is “large”} \quad .$$

---

[4] For any two vectors $\mathbf{a}$ and $\mathbf{b}$,

$$\|\mathbf{a} - \mathbf{b}\| = \text{distance between } \mathbf{a} \text{ and } \mathbf{b} = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2 + \cdots + (a_N - b_N)^2} \quad .$$

On the other hand, if you follow the trajectory passing through position $(x, y) = (-1, 0)$, then you probably get the impression that, as t increases, the trajectory is heading deeper into the third quadrant of the XY–plane, suggesting that if $\mathbf{x}(t) = [x(t), y(t)]^{\mathsf{T}}$ is any solution to the direction field's system satisfying $\mathbf{x}(t_0) = [-1, 0]^{\mathsf{T}}$ for some t_0, then

$$\lim_{t\to\infty} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -\infty \\ -\infty \end{bmatrix} \quad .$$

You may even suspect that, for such a solution,

$$y(t) \approx x(t) \qquad \text{when} \quad t \text{ is "large"} \quad ,$$

though there is hardly enough of the trajectory sketched to be too confident of this suspicion.

37.4 Existence and Uniqueness of Solutions and Trajectories

Existence and Uniqueness of Solutions

In section 3.3, two theorems were given — theorems 3.1 and 3.2 — describing conditions ensuring the existence and uniqueness of solutions to first-order differential equations. Here are the "systems versions" of those theorems:

Theorem 37.1 (existence and uniqueness for general systems)
Consider a first-order initial-value problem

$$\begin{bmatrix} {x_1}'(t) \\ {x_2}'(t) \\ \vdots \\ {x_N}'(t) \end{bmatrix} = \begin{bmatrix} f_1(t, x_1, x_2, \dots, x_N) \\ f_2(t, x_1, x_2, \dots, x_N) \\ \vdots \\ f_N(t, x_1, x_2, \dots, x_N) \end{bmatrix} \quad \text{with} \quad \begin{bmatrix} x_1(t_0) \\ x_2(t_0) \\ \vdots \\ x_N(t_0) \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{bmatrix} \quad .$$

Suppose each f_j and $\partial f_j/\partial x_k$ are continuous functions on an open region of the $TX_1X_2\cdots X_N$–space containing the point $(t_0, a_1, a_2, \dots, a_N)$. Then this initial-value problem has at least one solution. Moreover, there is an open interval (α, β) containing t_0 on which this is the only solution to this initial-value problem.

Theorem 37.2
Consider a first-order initial-value problem

$$\begin{bmatrix} {x_1}'(t) \\ {x_2}'(t) \\ \vdots \\ {x_N}'(t) \end{bmatrix} = \begin{bmatrix} f_1(t, x_1, x_2, \dots, x_N) \\ f_2(t, x_1, x_2, \dots, x_N) \\ \vdots \\ f_N(t, x_1, x_2, \dots, x_N) \end{bmatrix} \quad \text{with} \quad \begin{bmatrix} x_1(t_0) \\ x_2(t_0) \\ \vdots \\ x_N(t_0) \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{bmatrix} \quad .$$

Let (α, β) be an open interval containing t_0, and let $\mathcal{R}$ be the corresponding infinite strip

$$\mathcal{R} = \{ (t, x_1, x_2, \ldots, x_N) : \alpha < t < \beta \text{ and } -\infty < x_k < \infty \text{ for } k = 1, 2, \ldots, N \} \quad .$$

Assume that, on $\mathcal{R}$, each f_j and each ${}^{\partial f_j}/_{\partial x_k}$ are continuous functions with each ${}^{\partial f_j}/_{\partial x_k}$ also being a function of t only. Then this initial-value problem has exactly one solution over (α, β).

The proofs of the above two theorems are simply multidimensional versions of the proofs already discussed in sections 3.4, 3.5 and 3.6 for theorems 3.1 and 3.2.

Remember, though, that most of our systems will be regular, which means that the component functions all satisfy the conditions required in theorem 37.1. So we automatically have:

Corollary 37.3
Suppose $\mathbf{x}' = \mathbf{F}(t, \mathbf{x})$ is a regular system. Then any initial-value problem involving this system,

$$\mathbf{x}' = \mathbf{F}(t, \mathbf{x}) \qquad \text{with} \quad \mathbf{x}(t_0) = \mathbf{a} \quad ,$$

has at least one solution. Moreover, there is an open interval (α, β) containing t_0 on which this is the only solution to the given initial-value problem.

We should also note the following easily verified corollary of theorem 37.2. It will play a major role in the next chapter.

Theorem 37.4 (existence and uniqueness for linear systems)
Assume $\mathbf{P}$ is a continuous $N \times N$ matrix-valued function over the interval (α, β) and $\mathbf{g}$ is a continuous vector-valued function over (α, β), and let t_0 and $\mathbf{a}$ be, respectively, a point in (α, β) and a constant vector. Then the initial-value problem

$$\mathbf{x}' = \mathbf{P}\mathbf{x} + \mathbf{g} \qquad \text{with} \quad \mathbf{x}(t_0) = \mathbf{a} \quad ,$$

has exactly one solution over the interval (α, β).

?►*Exercise 37.1:* *Show that theorem 37.4 follows from theorem 37.2. Start by observing that*

$$\mathbf{x}' = \mathbf{P}\mathbf{x} + \mathbf{g} \qquad \text{with} \quad \mathbf{x}(t_0) = \mathbf{a} \quad ,$$

is the system given in theorem 37.2 with

$$f_k(t, x_1, x_2, \ldots, x_N) = p_{k1}(t)x_1 + p_{k2}(t)x_2 + \cdots + p_{kN}(t)x_N + g_k(t)$$

for $k = 1, 2, \ldots, N$.

Trajectories for Regular Autonomous Systems

We know that the trajectories of a regular autonomous system will "follow" the direction field of that system. Consequently, when we sketch oriented curves that "follow" a direction field for a given system, we naturally expect those curves to be trajectories for that system. But are we absolutely sure? Or could we have, say,

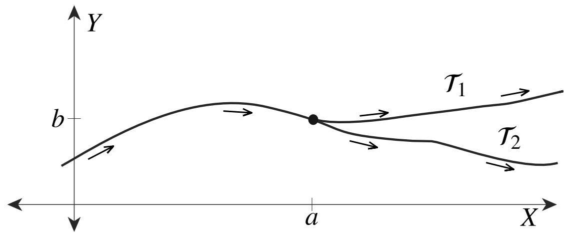

Figure 37.4: A trajectory "separating" into two different trajectories at (a, b) .

1. a trajectory occupying only part of the sketched curve, or

2. a trajectory that "splits off" of one of our curves, as in figure 37.4?

It turns out that the answer to each of the above is no, at least assuming our system is a regular autonomous system (and the sketched curve actually does "follow" the direction field perfectly). That is the gist of the following theorem and its corollaries.

Theorem 37.5
Assume $\mathbf{x}' = \mathbf{F}(\mathbf{x})$ is an $N \times N$ regular autonomous system, and let C be any oriented curve of nonzero length such that all the following hold at each point $\mathbf{x}$ in C :

1. *The point $\mathbf{x}$ is not a critical point for the system.*

2. *The curve C has a unique tangent line at $\mathbf{x}$, and that line is parallel to the vector $\mathbf{F}(\mathbf{x})$.*

3. *The direction of travel of C through $\mathbf{x}$ is in the same direction as given by the vector $\mathbf{F}(\mathbf{x})$.*

Then C (excluding any endpoints) is the trajectory for some solution

$$\mathbf{x}(t) = [x_1(t), x_2(t), \ldots, x_N(t)]^\mathsf{T} \qquad \text{for} \quad t_S < t < t_E$$

to the system $\mathbf{x}' = \mathbf{F}(\mathbf{x})$.

This theorem assures us that, in theory, the curves drawn "following" a direction field will be trajectories of our system (in practice, of course, the curves we actually draw will be approximations). Combining this thoerem with the existence and uniqueness results of corollary 37.3 leads to the next two corollaries regarding trajectories that are *maximal*; that is, trajectories that are not contained in any larger trajectories.

Corollary 37.6
Two different maximal trajectories of a regular autonomous system cannot intersect each other.

Corollary 37.7
If a maximal trajectory of a regular autonomous system has an endpoint, that endpoint must be a critical point.

We'll discuss the proof of the above theorem in the next section for those who are interested. Verifying the two corollaries will be left as exercises (see exercise 37.14).

37.5 Proving Theorem 37.5

The Assumptions

In all the following, let us assume we have some regular autonomous system of differential equations

$$\mathbf{x}' = \mathbf{F}(\mathbf{x}) \quad ,$$

along with an oriented curve C of nonzero length such that all the following hold at each point $\mathbf{x}$ in C :

1. The point $\mathbf{x}$ is not a critical point for the system.
2. The curve C has a unique tangent line at $\mathbf{x}$, and that line is parallel to the vector $\mathbf{F}(\mathbf{x})$.
3. The direction of travel of C through $\mathbf{x}$ is in the same direction as given by the vector $\mathbf{F}(\mathbf{x})$

Note that the requirement that C has a tangent line at each point in C means that we are excluding any endpoints of this curve.

For convenience, we will limit ourselves to curves on the XY–plane, though our analysis can easily be extend to higher dimensions.

Preliminaries

To verify our theorem, we will need some material that you should recall from your course on multivariable calculus.

Norms and Normalizations

The *norm* (or length) of a column vector or vector-valued function

$$\mathbf{v} = [v_1 \, , \, v_2]$$

is

$$\|\mathbf{v}\| = \sqrt{[v_1]^2 + [v_2]^2} \quad .$$

If $\mathbf{v}$ is a nonzero vector, then we can *normalize* it by dividing it by it's norm, obtaining a vector

$$\mathbf{n} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{\sqrt{[v_1]^2 + [v_2]^2}} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

of unit length (i.e., $\|\mathbf{n}\| = 1$) and pointing in the same direction as $\mathbf{v}$.

Oriented Curves and Unit Tangents

If $\mathbf{x}$ is any point on any oriented curve at which the curve has a well-defined tangent line, then this curve has a *unit tangent vector* at $\mathbf{x}$, denoted by $\mathbf{T}(\mathbf{x})$, which is simply a unit vector tangent to the curve at that point, and pointing in the direction of travel along the curve at that point. For our oriented curve, C , that tangent line is parallel to $\mathbf{F}(\mathbf{x})$ and the direction of travel is given by $\mathbf{F}(\mathbf{x})$. So the unit tangent at $\mathbf{x}$ must be the normalization of $\mathbf{F}(\mathbf{x})$. That is

$$\mathbf{T}(\mathbf{x}) = \frac{\mathbf{F}(\mathbf{x})}{\|\mathbf{F}(\mathbf{x})\|} \qquad \text{for each } \mathbf{x} \text{ in } C \quad . \tag{37.6}$$

Curve Parameterizations

A *parameterization* of an oriented curve C is an ordered pair of functions on some interval

$$(x(t), y(t)) \qquad \text{for} \quad t_S < t < t_E$$

that traces out the curve in the direction of travel along C as t varies from t_S to t_E.

Since our interest is in parametrizations given by the solutions to our system of differential equations, we will also write our parametrizations in column vector form,

$$\mathbf{x}(t) = [x(t)\,,\ y(t)]^{\mathsf{T}} \qquad \text{for} \quad t_S < t < t_E \quad .$$

Go ahead and view $\mathbf{x}(t)$ as the position of some moving object at time t. Then, provided the components are suitably differentiable,

$$\mathbf{x}'(t) = [x'(t)\,,\ y'(t)]^{\mathsf{T}}$$

is the corresponding "velocity", of the object at time t. This is a vector pointing in the direction of travel of the object at time t, and whose length,

$$\left\|\mathbf{x}'(t)\right\| = \sqrt{[x'(t)]^2 + [y'(t)]^2} \quad ,$$

is the speed of the object at time t (i.e., as it goes through position $(x(t), y(t))$). Recall that integrating this speed from $t = t_1$ to $t = t_2$,

$$\int_{t_1}^{t_2} \left\|\mathbf{x}'(t)\right\| \, dt \quad , \tag{37.7}$$

gives the signed distance one would travel along the curve to go from $(x(t_1), y(t_1))$ to $(x(t_2), y(t_2))$. This value is positive if $t_1 < t_2$ and negative if $t_1 > t_2$. Recall, also, that this distance (the "arclength") is traditionally denoted by s.

The most fundamental parametrizations are the arclength parametrizations. To define one for our oriented curve C, first pick some point (x_0, y_0) on C. Then let s_S and s_E be, respectively, the negative and positive values such that s_E is the "maximal distance" that can be traveled in the positive direction along C from (x_0, y_0), and $|s_S|$ is the "maximal distance" that can be traveled in the negative direction along C from (x_0, y_0). These distances may be infinite.[5]. Finally, define the arclength parametriation

$$\tilde{\mathbf{x}}(s) = [\tilde{x}(s)\,,\ \tilde{y}(s)] \qquad \text{for} \quad s_S < s < s_E$$

as follows (and as indicated in figure 37.5):

1. For $0 \le s < s_E$ set $(\tilde{x}(s), \tilde{y}(s))$ equal to the point on C arrived at by traveling in the positive direction along C by distance of s from (x_0, y_0).

2. For $s_S < s \le 0$ set $(\tilde{x}(s), \tilde{y}(s))$ equal to the point on C arrived at by traveling in the negative direction along C by distance of $|s|$ from (x_0, y_0).

We should note that if the curve intersects itself, then individual points may be given by $\tilde{\mathbf{x}}(s)$ for more than one value of s. In particular, if C is a loop of length L, then $\tilde{\mathbf{x}}$ will be periodic with $\tilde{\mathbf{x}}(s + L)$ for every real value s.

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[5] Better definitions for s_S and s_E are discussed in the 'technical note' at the end of this subsection

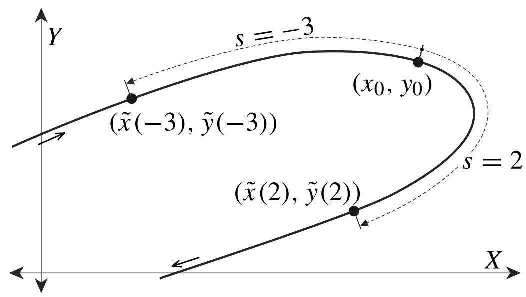


Figure 37.5: Two points given by an arclength parameterization $\tilde{\mathbf{x}}(s)$ of an oriented curve.

It should also be noted that, from the arclength integral (37.7) and the fact that, by definition, s is the signed distance one would travel along the curve to go from $(x(0), y(0))$ to $(x(s), y(s))$, we automatically have

$$\int_0^s \left\| \tilde{\mathbf{x}}'(\sigma) \right\| \, d\sigma = s \quad .$$

Differentiating this yields

$$\left\| \tilde{\mathbf{x}}'(s) \right\| = \frac{d}{ds} \int_0^s \left\| \tilde{\mathbf{x}}'(\sigma) \right\| \, d\sigma = \frac{ds}{ds} = 1 \quad .$$

Hence, each $\tilde{\mathbf{x}}'(s)$ is a unit vector pointing in the direction of travel on C at $\tilde{\mathbf{x}}(s)$ — that is, $\tilde{\mathbf{x}}'(s)$ is the unit tangent vector for C at $\tilde{\mathbf{x}}(s)$. Combining this with equation (37.6) yields

$$\tilde{\mathbf{x}}'(s) = \mathbf{T}(\mathbf{x}(s)) = \frac{\mathbf{F}(\mathbf{x})}{\|\mathbf{F}(\mathbf{x})\|} \qquad \text{for} \quad s_S < s < s_E \quad . \tag{37.8}$$

Technical Note on "Maximal Distances"

We set s_E equal to "the 'maximal distance' that can be traveled in the positive direction along C from (x_0, y_0)". Technically, this "maximal distance" may not exist because, technically, an endpoint of a trajectory need not actually be part of that trajectory.

To be more precise, let us define a subset $\mathcal{S}$ of the positive real numbers by specifying that

$$s \quad \text{is in} \quad \mathcal{S}$$

if and only

> there is a point on C arrived at by traveling a distance of s in the positive direction along C from (x_0, y_0).

With a little thought, it should be clear that $\mathcal{S}$ must be a subinterval of $(0, \infty)$ (assuming some 'obvious facts' about the nature of curves). One end point of $\mathcal{S}$ must clearly be 0. The other endpoint gives us the value s_E. In particular, letting C^+ be that part of C containing all the points arrived at by traveling in the positive direction along C from (x_0, y_0):

1. If C^+ is a closed loop, then $s_E = \infty$ (because we keep going around the loop as s increases).
2. If C^+ is a curve that does not intersect itself, then s_E is the length of C^+ (which may be infinite).

Obviously, similar comments can be made regarding the definition of s_S.

Finishing the Proof of Theorem 37.5

Let us now use the arclength parameterization $\tilde{\mathbf{x}}$ to define a function $\tilde{t}$ of s by

$$\tilde{t}(s) = \int_0^s \frac{1}{\|\mathbf{F}(\tilde{\mathbf{x}}(\sigma))\|} \, d\sigma$$

Since C contains no critical points, the integrand is always finite and positive, and the above function is a differentiable steadily increasing function with

$$\tilde{t}\,'(s) = \frac{1}{\|\mathbf{F}(\tilde{\mathbf{x}}(s))\|} \qquad \text{for} \quad s_S < s < s_E \quad .$$

Consequently, for each s in (s_E, s_E), there is exactly one corresponding t with $t = \tilde{t}(s)$. Thus, we can invert this relationship, defining a function $\tilde{s}$ by

$$\tilde{s}(t) = s \quad \Longleftrightarrow \quad t = \tilde{t}(s) \quad .$$

This function, $\tilde{s}$ is defined on the interval (t_S, t_E) where

$$t_S = \lim_{s \to s_S{}^+} \tilde{t}(s) \qquad \text{and} \qquad t_E = \lim_{s \to s_E{}^-} \tilde{t}(s) \quad .$$

By definition,

$$s = \tilde{s}\left(\tilde{t}(s)\right) \qquad \text{for} \quad s_S < s < s_E \quad .$$

From this, the chain rule and the above formula for $\tilde{t}\,'$, we get

$$1 = \frac{ds}{ds} = \frac{d}{ds}\left[\tilde{s}\left(\tilde{t}(s)\right)\right] = \tilde{s}\,'\left(\tilde{t}(s)\right)\tilde{t}\,'(s) = \tilde{s}\,'\left(\tilde{t}(s)\right) \cdot \frac{1}{\|\mathbf{F}(\tilde{\mathbf{x}}(s))\|} \quad .$$

Hence,

$$\tilde{s}\,'\left(\tilde{t}(s)\right) = \|\mathbf{F}(\tilde{\mathbf{x}}(s))\| \quad . \tag{37.9}$$

Now let

$$\mathbf{x}(t) = \tilde{\mathbf{x}}(\tilde{s}(t)) \qquad \text{for} \quad t_S < t < t_E \quad .$$

Observe that $\mathbf{x}(t)$ will trace out C as t varies from t_S to t_E, and that

$$\mathbf{F}(\mathbf{x}(t)) = \mathbf{F}\left(\tilde{\mathbf{x}}(\tilde{s}(t))\right) \quad .$$

This, along with the chain rule and equations (37.8) and (37.9), yields

$$\mathbf{x}'(t) = \frac{d}{dt}\left[\tilde{\mathbf{x}}(\tilde{s}(t))\right] = \tilde{\mathbf{x}}\,'(\tilde{s}(t)) \cdot \tilde{s}\,'(t) = \frac{\mathbf{F}(\mathbf{x}(t))}{\|\mathbf{F}(\mathbf{x}(t))\|} \cdot \|\mathbf{F}(\mathbf{x}(t)\| = \mathbf{F}(\mathbf{x}(t)) \quad ,$$

which finishes our proof of theorem 37.5. ∎

37.6 Existence and Uniqueness for Single N^{th}-order Differential Equations*

Several theorems regarding the existence and uniqueness of solutions to a single second- or higher-order differential equation were given near the end of chapter 11. It is worth noting that all of these theorems can be derived from the results given in this chapter for the existence and uniqueness of solutions to a system of differential equations. To see this, let us consider a single N^{th}-order initial-value problem of the form

$$y^{(N)} = F\left(t, y, y', \dots, y^{(N-1)}\right) \tag{37.10a}$$

with

$$y(t_0) = a_1 \quad , \quad y'(t_0) = a_2 \quad , \quad \dots \quad \text{and} \quad y^{(n-1)}(t_0) = a_N \quad . \tag{37.10b}$$

In this:

1. We are using t as the basic variable (so $y = y(t)$ and $y^{(k)} = {}^{d^k y}/_{dt^k}$).
2. t_0 , a_1 , a_1 , ... and a_N are fixed values.
3. $F(t, x_1, x_2, \dots, x_N)$ is a function of $N + 1$ variables on some open region $\mathcal{R}$ of $TX_1X_2 \cdots X_N$–space containing the point $(t_0, a_1, a_2, \dots, a_N)$.

Our goal will be to derive theorem 11.3 on page 289. So assume, as in that theorem, that F and each ${}^{\partial F}/_{\partial x_k}$ is continuous on the region $\mathcal{R}$.

Let us now convert this single differential equation to a standard first-order system by introducing N new unknown functions x_1 , x_2 , ... and x_N related to y via

$$x_1 = y \quad , \quad x_2 = y' \quad , \quad x_3 = y'' \quad , \quad \dots \quad \text{and} \quad x_N = y^{(N-1)} \quad .$$

Then we have

$$\begin{aligned}
{x_1}' &= y' = x_2 \quad , \\
{x_2}' &= \left(y'\right)' = y'' = x_3 \quad , \\
{x_3}' &= \left(y''\right)' = y''' = x_4 \quad , \\
&\vdots \\
{x_{N-1}}' &= \left(y^{(N-2)}\right)' = y^{(N-1)} = x_N \quad ,
\end{aligned}$$

and, finally,

$${x_N}' = \left(y^{(N-1)}\right)' = y^{(N)} = F\left(t, y, y', \dots, y^{(n-1)}\right) = F(t, x_1, x_2, x_3, \dots, x_N) \quad .$$

Thus, our original initial-value problem can be rewritten as

$$\begin{bmatrix} {x_1}'(t) \\ {x_2}'(t) \\ \vdots \\ {x_N}'(t) \end{bmatrix} = \begin{bmatrix} f_1(t, x_1, x_2, \dots, x_N) \\ f_2(t, x_1, x_2, \dots, x_N) \\ \vdots \\ f_N(t, x_1, x_2, \dots, x_N) \end{bmatrix} \quad \text{with} \quad \begin{bmatrix} x_1(t_0) \\ x_2(t_0) \\ \vdots \\ x_N(t_0) \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{bmatrix} \tag{37.11}$$

* The material in this section plays no role in later developments, and can be safely skipped by those more interested in learning more about systems of differential equations than in verifying existence theorems for single equations.

where

$$f_1(t, x_1, x_2, \ldots, x_N) = x_2 \quad ,$$
$$f_2(t, x_1, x_2, \ldots, x_N) = x_3 \quad ,$$
$$f_3(t, x_1, x_2, \ldots, x_N) = x_4 \quad ,$$
$$\vdots$$

and

$$f_{N-1}(t, x_1, x_2, \ldots, x_N) = x_N \quad ,$$

while

$$f_N(t, x_1, x_2, \ldots, x_N) = F(t, x_1, x_2, x_3, \ldots, x_N) \quad .$$

It is almost trivial to verify that each f_j and each $\partial f_j/\partial x_k$ with $j \neq N$ is a continuous function on the region $\mathcal{R}$. Moreover, the assumptions made on F ensure that f_N and the $\partial f_N/\partial x_k$'s are also continuous functions on $\mathcal{R}$. This means theorem 37.1 on page 37–18 applies and tells us that

1. initial-value problem (37.11) has a solution $\mathbf{x}(t) = [x_1(t), x_2(t), \ldots, x_N(t)]^{\mathsf{T}}$.

and

2. there is an open interval (a, b) containing t_0 on which this $\mathbf{x}$ is the only solution to this initial-value problem.

Setting $y(t) = x_1(t)$ then gives us a solution to the original initial-value problem, problem (37.10), valid at least on the interval (a, b). Moreover, it must be the only solution on this interval because if there were a second solution, call it $\widehat{y}(t)$, then

$$\widehat{\mathbf{x}}(t) = \left[\widehat{y}(t), \widehat{y}'(t), \ldots, \widehat{y}^{(N-1)}(t)\right]^{\mathsf{T}}$$

would be a second solution to initial-value problem (37.11) (contrary to what we already know).

In summary, we've just used material developed for systems of differential equations to verify the following theorem:

Theorem 37.8 (existence and uniqueness for N^{th}-order initial-value problems)
Let t_0, a_1, a_2, … and a_N be fixed values, and let $F = F(t, x_1, x_2, \ldots, x_N)$ be some function of $N + 1$ variables. Assume F and the partial derivatives

$$\frac{\partial F}{\partial x_1} \quad , \quad \frac{\partial F}{\partial x_2} \quad , \quad \ldots \quad \text{and} \quad \frac{\partial F}{\partial x_N}$$

are all continuous functions in some open region containing the point $(t_0, a_1, a_2, \ldots, a_N)$. Then the initial-value problem

$$y^{(N)} = F\left(t, y, y', \ldots, y^{(N-1)}\right)$$

with

$$y(t_0) = a_1 \quad , \quad y'(t_0) = a_2 \quad , \quad \ldots \quad \text{and} \quad y^{(n-1)}(t_0) = a_N \quad .$$

has at least one solution $y = y(x)$. Moreover, there is an open interval (α, β) containing t_0 on which this y is the only solution to this initial-value problem.

If you now go back and compare the above theorem with theorem 11.3 on page 289, you will find that (except for cosmetic differences in notation) the two theorems are the same.

Additional Exercises

37.2. *Rewrite each of the following linear systems of differential equations in matrix/vector form.*

a. $\begin{aligned} x' &= 3x + 5y \\ y' &= -5x + 7y \end{aligned}$

b. $\begin{aligned} {x_1}' &= x_2 \\ {x_2}' &= x_3 \\ {x_3}' &= 4x_1 + 3x_2 - 2x_3 \end{aligned}$

c. $\begin{aligned} x' &= 2x - y + 2z + 4 \\ y' &= 2y - 4z + 5 \\ z' &= 9x - 3z + 6 \end{aligned}$

d. $\begin{aligned} {x_1}' &= 2x_2 - t^2 x_3 + \sin(t) \\ {x_2}' &= (t+1)x_1 + t x_3 - \cos(t) \\ {x_3}' &= 3t^3 x_2 + \sqrt{t} \end{aligned}$

37.3. *Find all the constant/equilibrium solutions to each of the following systems:*

a. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 & -5 \\ 3 & -7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

b. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 & -5 \\ 3 & -7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 4 \\ 5 \end{bmatrix}$

c. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 6 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

d. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} xy - 6y \\ x - y - 5 \end{bmatrix}$

e. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x^2 - y^2 \\ x^2 - 6x + 8 \end{bmatrix}$

f. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \sin(y) \\ x^2 - 6x + 9 \end{bmatrix}$

g. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 4x - xy \\ x^2 y + y^3 - x^2 - y^2 \end{bmatrix}$

h. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x^2 + y^2 + 4 \\ 2x - 6y \end{bmatrix}$

37.4. *Sketch direction arrows for the system*

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -x + 2y \\ 2x - y \end{bmatrix}$$

at the points $(x, y) = (2, 0)$, $(x, y) = (2, 2)$ *and* $(x, y) = (0, 2)$.

37.5. *Sketch the direction field for the system*

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} (2x - 1)(y + 1) \\ y - x \end{bmatrix}$$

on the 3×3 *grid with* $x = 0$, $1/2$ *and* 1 , *and with* $y = 0$, $1/2$ *and* 1 .

37.6. *A direction field for*

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

has been sketched to the right. Using this system and direction field:

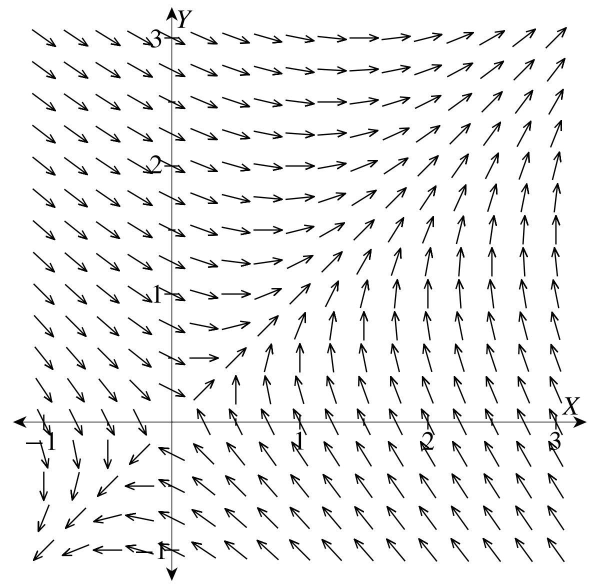
a. *Find and plot all the critical points.*

b i. *Sketch the trajectories that go through points* $(1, 0)$ *and* $(0, 1)$.

ii. *Sketch a phase portrait for this system.*

iii. *Suppose* $[x(t), y(t)]^{\mathsf{T}}$ *is the solution to this system satisfying* $[x(0), y(0)]^{\mathsf{T}} = [1, 0]$. *What apparently happens to* $x(t)$ *and* $y(t)$ *as* t *gets large?*

c. *As well as you can, decide whether the critical point found above is asymptotically stable, stable but not asymptotically stable, or unstable.*



37.7. *A direction field for*

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

has been sketched to the right. Using this system and direction field:

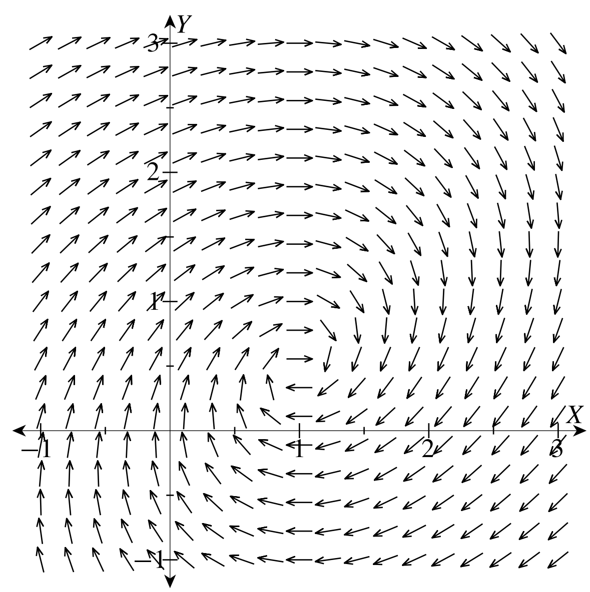
a. *Find and plot all the critical points.*

b i. *Sketch the trajectories that go through points* $(-1, 0)$ *and* $(0, 2)$.

ii. *Sketch a phase portrait for this system.*

iii. *Suppose* $[x(t), y(t)]^{\mathsf{T}}$ *is the solution to this system satisfying* $[x(0), y(0)]^{\mathsf{T}} = [-1, 0]$. *What apparently happens to* $x(t)$ *and* $y(t)$ *as* t *gets large?*

c. *As well as you can, decide whether the critical point found in part* ***a*** *is asymptotically stable, stable but not asymptotically stable, or unstable.*



37.8. A direction field for

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} y \\ -\sin(2x) \end{bmatrix}$$

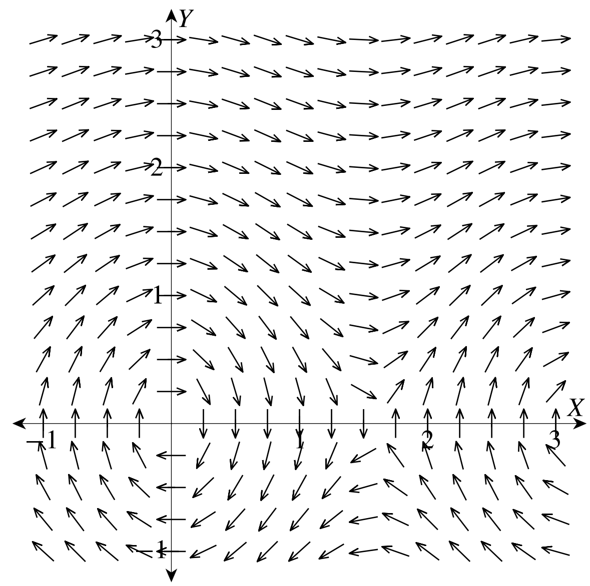
has been sketched to the right. Using this system and direction field:

a. Find and plot all the critical points.

b i. Sketch the trajectories that go through points $(1,0)$ and $(0,2)$.

ii. Suppose $[x(t), y(t)]^{\mathsf{T}}$ is the solution to this system satisfying $[x(0), y(0)]^{\mathsf{T}} = [1, 0]$. What apparently happens to $x(t)$ and $y(t)$ as t gets large?

c. All the critical points of this system are either stable (but not asymptotically stable) or unstable. Using this direction field, determine which are stable, and which are unstable.



37.9. A direction field for

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -1 \\ -2 \end{bmatrix}$$

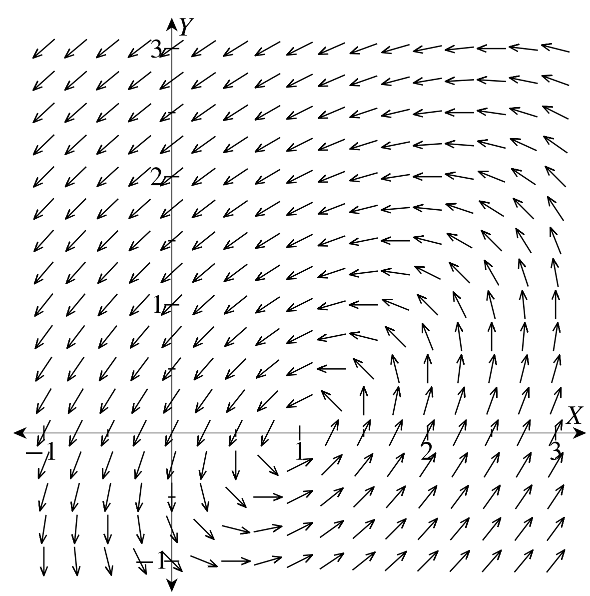
has been sketched to the right. Using this system and direction field:

a. Find and plot all the critical points.

b i. Sketch the trajectories that go through points $(0,0)$ and $(0,1)$.

ii. Suppose $[x(t), y(t)]^{\mathsf{T}}$ is the solution to this system satisfying $[x(0), y(0)]^{\mathsf{T}} = [0, 0]$. What apparently happens to $x(t)$ and $y(t)$ as t gets large?

c. As well as you can, decide whether the critical point found in part ***a*** is asymptotically stable, stable but not asymptotically stable, or unstable.



37.10. *A direction field for*

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ -1 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

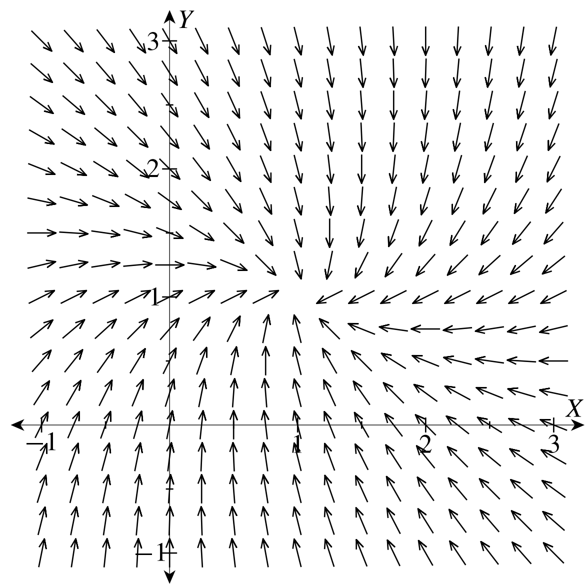
has been sketched to the right. Using this system and direction field:

a. *Find and plot all the critical points.*

b i. *Sketch the trajectories that go through points* $(2, 0)$ *and* $(0, 2)$.

ii. *Suppose* $[x(t), y(t)]^{\mathsf{T}}$ *is the solution to this system satisfying* $[x(0), y(0)]^{\mathsf{T}} = [1, 0]$. *What apparently happens to* $x(t)$ *and* $y(t)$ *as* t *gets large?*

c. *As well as you can, decide whether the critical point found in part* ***a*** *is asymptotically stable, stable but not asymptotically stable, or unstable.*



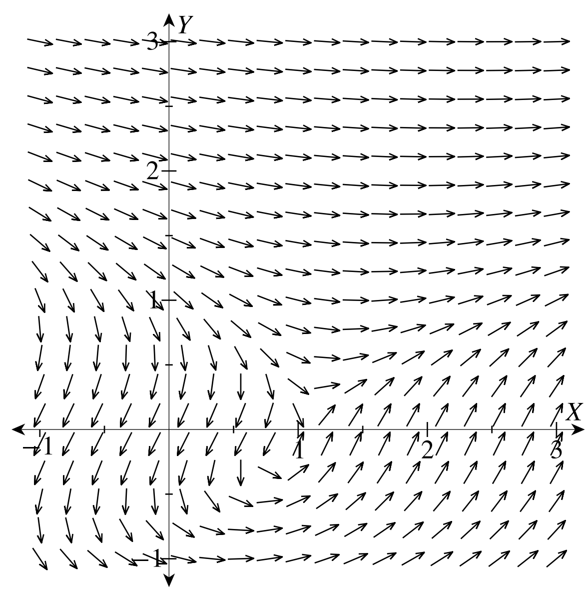
37.11. *A direction field for*

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x + 4y^2 - 1 \\ 2x - y - 2 \end{bmatrix}$$

has been sketched to the right. Using this system and direction field:

a. *Find and plot all the critical points.*

b. *Sketch the trajectories that go through the points* $(0, 0)$ *and* $(1, 1)$.



37.12. *Look up the commands for generating direction fields for systems of differential equations in your favorite computer math package. Then, use these commands to do the following for each problem below:*

i. *Sketch the indicated direction field for the given system.*

ii. *Use the resulting direction field to sketch (by hand) a phase portrait for the system.*

a. *The system is*

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -x + 2y \\ 2x - y \end{bmatrix} \quad .$$

Use a 25×25 *grid on the region* $-1 \le x \le 3$ *and* $-1 \le y \le 3$ *. (Compare the resulting direction field to the direction arrows computed in exercise 37.4.)*

b. *The system is*

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} (2x - 1)(y + 1) \\ y - x \end{bmatrix} \quad .$$

Use a 25×25 *grid on the region* $-1 \le x \le 3$ *and* $-1 \le y \le 3$ *. (Compare the resulting direction field to the direction field found in exercise 37.5.)*

c. *The system is*

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x + 4y^2 - 1 \\ 2x - y - 2 \end{bmatrix} \quad .$$

Use a 25×25 *grid on the region* $0 \le x \le 2$ *and* $-1 \le y \le 1$ *. (This gives a 'close up' view of the critical points of the system in exercise 37.11.)*

d. *The system is*

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x + 4y^2 - 1 \\ 2x - y - 2 \end{bmatrix} \quad .$$

Use a 25×25 *grid on the region* $3/4 \le x \le 5/4$ *and* $-1/4 \le y \le 1/4$ *. (An even 'closer up' view of the critical points of the system in exercise 37.11.)*

37.13. *Let* $\tilde{\mathbf{x}}(t)$ *be a vector-valued function on the interval* (α, β) *, and assume* $\tilde{\mathbf{x}}$ *is a solution to some initial-value problem*

$$\mathbf{x}' = \mathbf{F}(\mathbf{x}) \qquad \text{with} \quad \mathbf{x}(t_0) = \mathbf{a}$$

where $\mathbf{x}' = \mathbf{F}(\mathbf{x})$ *is a regular autonomous system and* t_0 *is some point in the interval* (α, β) *. Verify that a solution to*

$$\mathbf{x}' = \mathbf{F}(\mathbf{x}) \qquad \text{with} \quad \mathbf{x}(t_1) = \mathbf{a}$$

is then given by

$$\mathbf{x}(t) = \tilde{\mathbf{x}}(t - \tau) \qquad \text{for} \quad \alpha - \tau < t < \beta - \tau$$

where $\tau = t_1 - t_0$ *.*

37.14. *Using corollary 37.3 on page 37–19 on the existence and uniqueness of solutions to regular systems and theorem 37.5 on page 37–20 on curves being trajectories, along with (possibly) the results of exercise 37.13, verify the following:*

a. *Corollary 37.6 on page 37–20. (Hint: Start by assuming the two trajectories do intersect.)*

b. *Corollary 37.7 on page 37–20. (Hint: Start by assuming an endpoint of a given maximal trajectory is not a critical point.)*

Some Answers to Some of the Exercises

WARNING! Most of the following answers were prepared hastily and late at night. They have not been properly proofread! Errors are likely!

2a. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ -5 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

2b. $\begin{bmatrix} {x_1}' \\ {x_2}' \\ {x_3}' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & 3 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

2c. $\begin{bmatrix} {x_1}' \\ {x_2}' \\ {x_3}' \end{bmatrix} = \begin{bmatrix} 2 & -1 & 2 \\ 0 & 2 & -4 \\ 9 & 0 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$

2d. $\begin{bmatrix} {x_1}' \\ {x_2}' \\ {x_3}' \end{bmatrix} = \begin{bmatrix} 0 & 2 & -t^2 \\ t+1 & 0 & t \\ 0 & 3t^3 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} \sin(t) \\ -\cos(t) \\ \sqrt{t} \end{bmatrix}$

3a. $[x\,,\ y]^{\mathsf{T}} = [0\,,\ 0]^{\mathsf{T}}$

3b. $[3\,,\ 2]^{\mathsf{T}}$

3c. every $[x_0\,,\ y_0]^{\mathsf{T}}$ with $y_0 = -3x_0$

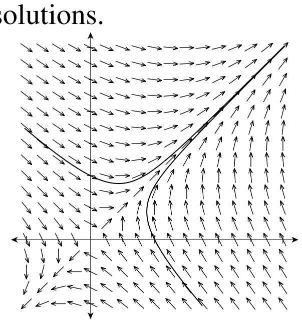
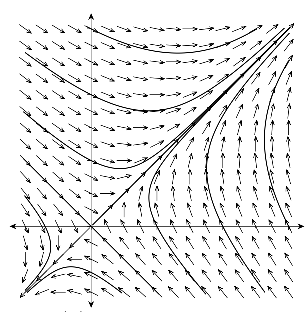
3d. $[6\,,\ 1]^{\mathsf{T}}$ and $[5\,,\ 0]^{\mathsf{T}}$

3e. $[2\,,\ 2]^{\mathsf{T}}$, $[2\,,\ -2]^{\mathsf{T}}$, $[4\,,\ 4]^{\mathsf{T}}$ and $[4\,,\ -4]^{\mathsf{T}}$

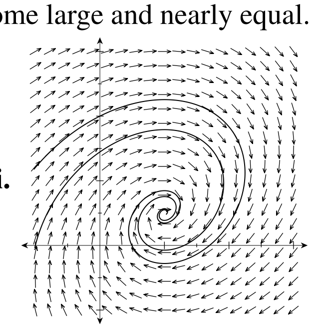
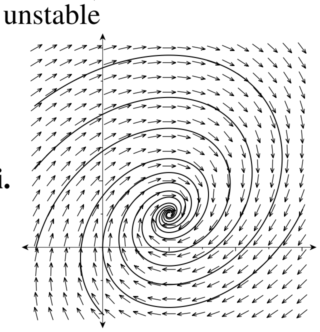
3f. $[3\,,\ n\pi]^{\mathsf{T}}$ for $n = 0, \pm 1, \pm 2, \pm 3, \ldots$

3g. $[0\,,\ 0]^{\mathsf{T}}$ and $[0\,,\ 1]^{\mathsf{T}}$

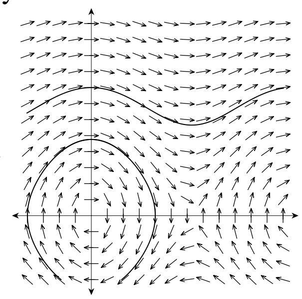
3h. No constant solutions.

6. a. $(0, 0)$, **bi.**  , **bii.**  ,

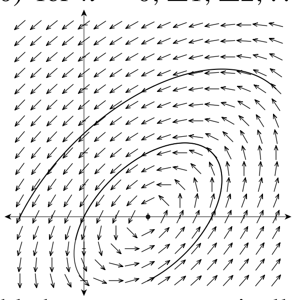
biii. They become large and nearly equal. , **c.** unstable

7. a. $(1, 1/2)$, **bi.**  , **bii.**  ,

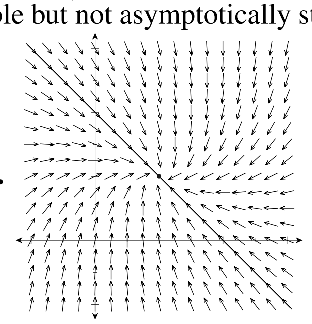
biii. $(x, y) \to (1, {}^1\!/_2)$, **c.** asymptotically stable

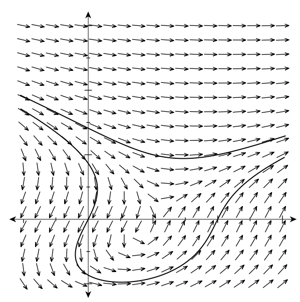
8. a. $({}^{n\pi}\!/_2, 0)$ for $n = 0, \pm 1, \pm 2, \ldots$, **bi.**  , **bii.** (x, y) "orbits" about $(0, 0)$ clockwise.

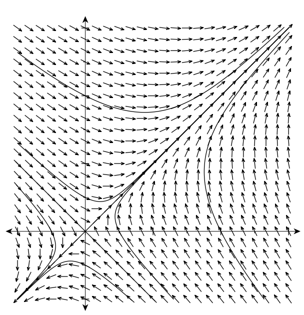
, **c.** stable: $(n\pi, 0)$ for $n = 0, \pm 1, \pm 2, \ldots$; unstable: $({}^{n\pi}\!/_2, 0)$ for $n = 1, \pm 3, \pm 5, \ldots$

9. a. $(1, 0)$, **bi.**  , **bii.** (x, y) "orbits" about $(x, 0)$ counterclockwise.

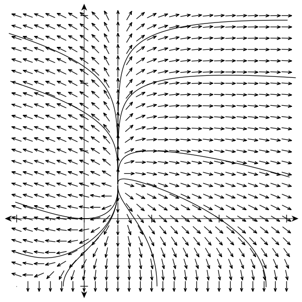
, **c.** (probably) stable but not asymptotically stable

10. a. $(1, 1)$, **bi.**  , **bii.** $(x, y) \to (1, 1)$, **c.** asymptotically stable

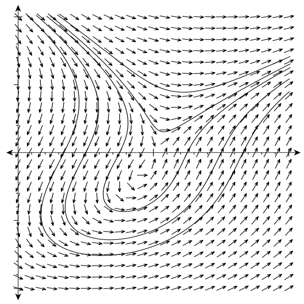
11. a. $(1, 0)$ and $({}^{15}\!/_{16}, -{}^1\!/_8)$, **bi.** 

12a. 

12b.



12c.



12d.