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Series Solutions: Preliminaries (A Brief Review of Infinite Series, Power Series and a Little Complex Variables)

At this point, you should have no problem in solving any differential equation of the form

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0 \qquad \text{or} \qquad ax^2\frac{d^2y}{dx^2} + bx\frac{dy}{dx} + cy = 0$$

when a, b and c are all constants. You've even solved a few (in chapters 11 and 13) in which a, b and/or c where not constants. Unfortunately, the methods used in those chapters are somewhat limited. More general methods do exist, and, in the next few chapters, we will discuss some of the more important of these in which solutions are described in terms of "power series" and "modified power series".

Ideally, you are already well-enough acquainted with infinite series and power series to jump right into the discussion of the next chapter. As a precaution, though, you may want to skim through this chapter. It is a brief review of infinite series with an emphasis on power series, along with a brief discussion of using complex variables in these series. As much as possible, we'll limit our discussion to topics that will be needed in the next few chapters, including a few that probably were not emphasized during your first exposure to power series.

30.1 Infinite Series

Basic Basics

Recall that, in the language of mathematics, an *infinite series* is a summation with infinitely many terms. For example,

$$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$$

is the infamous *harmonic series*. More generally, an infinite series is anything of the form

$$\sum_{k=\gamma}^{\infty} \alpha_k \qquad \text{or (equivalently)} \qquad \alpha_\gamma + \alpha_{\gamma+1} + \alpha_{\gamma+2} + \alpha_{\gamma+3} + \alpha_{\gamma+4} + \cdots$$

where γ, the starting index, is some integer (often, it's 0 or 1), and the α_k's are things that can be added together. These α_k's may be numbers, functions or even matrices. For the moment, we will assume them to be numbers (as in the harmonic series, above).

Given an arbitrary infinite series

$$\sum_{k=\gamma}^{\infty} \alpha_k = \alpha_\gamma + \alpha_{\gamma+1} + \alpha_{\gamma+2} + \alpha_{\gamma+3} + \alpha_{\gamma+4} + \cdots$$

and any integer $N \geq \gamma$, we define the corresponding N^{th} *partial sum* S_N by[1]

$$\begin{aligned} S_N &= \text{sum of all terms from } a_\gamma \text{ to } a_N \\ &= \sum_{k=\gamma}^{N} \alpha_k = \alpha_\gamma + \alpha_{\gamma+1} + \alpha_{\gamma+2} + \cdots + \alpha_N \quad . \end{aligned}$$

Observe that

$$\lim_{N\to\infty} S_N = \lim_{N\to\infty} \sum_{k=\gamma}^{N} \alpha_k = \sum_{k=\gamma}^{\infty} \alpha_k = \alpha_\gamma + \alpha_{\gamma+1} + \alpha_{\gamma+2} + \alpha_{\gamma+3} + \alpha_{\gamma+4} + \cdots \quad .$$

Naturally, the usefulness of an infinite series usually depends on whether it actually adds up to some finite value; that is, whether

$$\lim_{N\to\infty} \sum_{k=\gamma}^{N} \alpha_k$$

is some finite value. If the above limit does exist as a finite value, then we say our series *converges*, and call that value the *sum* of that series (freely using $\sum_{k=\gamma}^{\infty} \alpha_k$ to denote both the series and its sum). On the other hand, if this limit does not exist as a finite value, then we say the series *diverges*.

Recall the following simple facts:

1. If $\sum_{k=\gamma}^{\infty} \alpha_k$ converges, then we can closely approximate its sum by any N^{th} partial sum, provided we choose N large enough.

2. If $\sum_{k=\gamma}^{\infty} \alpha_k$ converges, then its terms must shrink to zero as k gets large,

$$\alpha_k \rightarrow 0 \qquad \text{as} \qquad k \rightarrow \infty \quad .$$

Consequently, $\sum_{k=\gamma}^{\infty} \alpha_k$ can*not* converge (i.e., must diverge) if the terms do not shrink to zero as k gets large.

However, it is quite possible to have a series $\sum_{k=\gamma}^{\infty} \alpha_k$ that diverges even though

$$\alpha_k \rightarrow 0 \qquad \text{as} \qquad k \rightarrow \infty \quad .$$

The harmonic series, above, is one example. Even though

$$\alpha_k = \frac{1}{k} \rightarrow 0 \qquad \text{as} \qquad k \rightarrow \infty \quad ,$$

you can easily show that the series diverges (to infinity) using the integral test.

[1] It's also standard to define S_N to be the sum of the first N terms. Our choice will be slightly more convenient.

3. If $\sum_{k=\gamma}^{\infty} \alpha_k$ and $\sum_{k=\gamma}^{\infty} \beta_k$ are both convergent series, and A and B are any two finite numbers, then the series $\sum_{k=\gamma}^{\infty}[A\alpha_k]$ and $\sum_{k=\gamma}^{\infty}[A\alpha_k + B\beta_k]$ are also convergent. Moreover,

$$\sum_{k=\gamma}^{\infty}[A\alpha_k] = A\sum_{k=\gamma}^{\infty}\alpha_k \qquad \text{and} \qquad \sum_{k=\gamma}^{\infty}[A\alpha_k + B\beta_k] = A\sum_{k=\gamma}^{\infty}\alpha_k + B\sum_{k=\gamma}^{\infty}\beta_k \quad .$$

To illustrate some of the above concepts, and to give us a first glimpse of "power series", let's look at the "geometric series".

The Geometric Series

Let x be any finite value and let γ be any nonnegative integer. The corresponding *geometric series* is

$$\sum_{k=\gamma}^{\infty} x^k = x^\gamma + x^{\gamma+1} + x^{\gamma+2} + x^{\gamma+3} + x^{\gamma+4} + \cdots \quad .$$

If $\gamma = 0$, we may refer to this as a *basic* geometric series.[2]

Letting $\gamma = 0$ and using, respectively, $x = 0$, $1/2$, $-1/2$, 1, -1, 2, and -2 gives us the following geometric series:

$$\sum_{k=0}^{\infty} 0^k = 1 + 0 + 0 + 0 + 0 + \cdots \quad ,$$

$$\sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots \quad ,$$

$$\sum_{k=0}^{\infty} \left(-\frac{1}{2}\right)^k = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \cdots \quad ,$$

$$\sum_{k=0}^{\infty} 1^k = 1 + 1 + 1 + 1 + 1 + \cdots \quad ,$$

$$\sum_{k=0}^{\infty} (-1)^k = 1 - 1 + 1 - 1 + 1 - \cdots \quad ,$$

$$\sum_{k=0}^{\infty} 2^k = 1 + 2 + 4 + 8 + 16 + \cdots \quad ,$$

and

$$\sum_{k=0}^{\infty} (-2)^k = 1 - 2 + 4 - 8 + 16 - \cdots \quad .$$

[2] Since 0^0 is an indeterminant form, it may be argued that there is a problem with the x^0 term when $x = 0$. However, in every geometric series with $x \neq 0$, $x^0 = 1$. So, to be consistent, we automatically assume $x^0 = 1$ in a geometric series when $x = 0$.

It should be obvious that a geometric series $\sum_{k=0}^{\infty} x^k$ converges if $x = 0$ and diverges whenever $|x| \geq 1$. It will also be worth noting that

$$\sum_{k=\gamma}^{\infty} x^k = x^{\gamma} + x^{\gamma+1} + x^{\gamma+2} + x^{\gamma+3} + x^{\gamma+4} + \cdots$$

$$= \sum_{k=0}^{\infty} x^{\gamma+k} = \sum_{k=0}^{\infty} x^{\gamma} x^k = x^{\gamma} \sum_{k=0}^{\infty} x^k \quad .$$

That is,

$$\sum_{k=\gamma}^{\infty} x^k = x^{\gamma} \sum_{k=0}^{\infty} x^k \quad . \tag{30.1}$$

Geometric series are unusual in that rather simple formulas can be derived for their partial sums. To see this, let

$$S_N = \sum_{k=0}^{N} x^k = x^0 + x^1 + x^2 + \cdots + x^N \quad .$$

If $x = 1$, then

$$S_N = \sum_{k=0}^{N} 1^k = \underbrace{1 + 1 + 1 + \cdots + 1}_{N+1 \text{ terms}} = N + 1 \quad .$$

If $x \neq 1$, then

$$\begin{aligned}
(1-x)S_N &= S_N - xS_N \\
&= \left[x^0 + x^1 + x^2 + \cdots + x^N\right] \\
&\qquad - x\left[x^0 + x^1 + x^2 + \cdots + x^N\right] \\
&= \left[1 + x^1 + x^2 + \cdots + x^N\right] \\
&\qquad - \left[x^1 + x^2 + x^3 + \cdots + x^{N+1}\right] \\
&= 1 - x^{N+1} \quad .
\end{aligned}$$

Dividing by $1 - x$ then gives us

$$\sum_{k=0}^{N} x^k = S_N = \frac{1 - x^{N+1}}{1 - x} \qquad \text{for} \quad x \neq 1 \quad . \tag{30.2}$$

!►*Example 30.1:* *With $x = {}^1\!/_2$, the above formula for S_N becomes*

$$S_N = \frac{1 - \left(\frac{1}{2}\right)^{N+1}}{1 - \frac{1}{2}} = 2\left[1 - \left(\frac{1}{2}\right)^{N+1}\right] \quad .$$

In particular,

$$S_4 = 2\left[1 - \left(\frac{1}{2}\right)^{4+1}\right] = 2\left[1 - \frac{1}{32}\right] = \frac{31}{16} \quad .$$

Of greater interest is that

$$\lim_{N\to\infty} S_N = \lim_{N\to\infty} 2\left[1 - \left(\frac{1}{2}\right)^{N+1}\right] = 2\left[1 - \lim_{N\to\infty}\left(\frac{1}{2}\right)^{N+1}\right] = 2[1-0] = 2 \quad .$$

Thus, the geometric series with $x = 1/2$ *converges, and*

$$\sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^k = \lim_{N\to\infty} S_N = 2 \quad .$$

?► ***Exercise 30.1:*** *Repeat the computations done in the last example, but using* $x = -1/2$ *. Show that the corresponding geometric series converges with*

$$\sum_{k=0}^{\infty}\left(-\frac{1}{2}\right)^k = \frac{2}{3} \quad .$$

As you can easily verify for yourself (and as illustrated in the above example and exercise),

$$\lim_{N\to\infty} x^{N+1} = 0 \qquad \text{whenever} \quad |x| < 1 \quad .$$

This, along with equations (30.2) and (30.1) (and some of the other comments above), leads to the following:

Theorem 30.1 (geometric series)
The basic geometric series $\sum_{k=0}^{\infty} x^k$ *converges if* $|x| < 1$ *and diverges if* $|x| \geq 1$ *. Moreover,*

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \qquad \textit{for} \quad |x| < 1 \quad .$$

More generally, for any nonnegative integer γ *, the geometric series* $\sum_{k=\gamma}^{\infty} x^k$ *converges if and only if* $|x| < 1$ *. Moreover,*

$$\sum_{k=\gamma}^{\infty} x^k = \frac{x^\gamma}{1-x} \qquad \textit{for} \quad |x| < 1 \quad .$$

Absolute Convergence and Convergence Tests

Absolute and Conditional Convergence

Recall that a series $\sum_{k=\gamma}^{\infty} \alpha_k$ can converge "absolutely" or "conditionally". It *converges absolutely* if and only if the corresponding series of absolute values

$$\sum_{k=\gamma}^{\infty} |\alpha_k|$$

converges. Basically, in an absolutely convergent series, the terms are shrinking to zero fast enough as the index increases to ensure convergence. Consequently, it's easily verified that an absolutely convergent series is, as the terminology suggests, convergent. Moreover, by repeatedly using the triangle inequality,

$$|a+b| \le |a| + |b| \quad ,$$

you can easily verify that

$$\left|\sum_{k=\gamma}^{\infty} \alpha_k\right| \le \sum_{k=\gamma}^{\infty} |\alpha_k| \quad .$$

If a series converges but is not absolutely convergent, then it is converging because each term "cancels out" some of the previous terms, and the series is said to be *conditionally convergent*. Such a convergence is somewhat unstable, and can be upset by, say, rearranging the terms of the series in a clever way. Because of this, we will much prefer series that converge absolutely.

Tests for Convergence and Divergence

In practice, it is rarely possible to determine the convergence of an infinite series by using its partial sums, simply because it is rarely possible find usable formulas for these partial sums. That is why, in your calculus course, you were exposed to several "tests" for determining whether a given series converges or diverges. One, of course, is the basic comparison test.

Theorem 30.2 (the comparison test)
Let $\sum_{k=\gamma}^{\infty} \alpha_k$ and $\sum_{k=\mu}^{\infty} \beta_k$ be two infinite series of real numbers, and suppose that, for some integer K ,

$$0 \le \alpha_k \le \beta_k \qquad \text{whenever} \quad K \le k \quad .$$

Then

$$\sum_{k=\mu}^{\infty} \beta_k \ \text{converges} \quad \Longrightarrow \quad \sum_{k=\gamma}^{\infty} \alpha_k \ \text{converges absolutely} \quad ,$$

while

$$\sum_{k=\gamma}^{\infty} \alpha_k \ \text{diverges} \quad \Longrightarrow \quad \sum_{k=\mu}^{\infty} \beta_k \ \text{diverges} \quad .$$

We'll be using the above test in a few pages, and will briefly discuss two other well-known tests (the limit comparison and the limit ratio tests) later.

There are, of course, many other "tests for convergence", including the alternating series test, the integral test, the basic ratio test, and the root test. I'm sure you remember them all fondly, and will be disappointed to learn that we will find little use for these other tests.

30.2 Power Series and Analytic Functions

Definition and Examples

A *power series* is any series of the form

$$\sum_{k=\gamma}^{\infty} a_k (x-x_0)^k$$

where x is a variable, x_0 and the a_k's are constants, and the starting index, γ, is some nonnegative integer. We'll often refer to x_0 as the *center* of the series, and say that the above power series is *centered at* or *about* x_0. We will also refer to the term $a_k(x-x_0)^k$ as the k^{th}*-order term* of the series.[3]

In theory, γ can be any nonnegative integer; in practice, γ is often 0. Even when $\gamma \neq 0$, we can assume

$$\sum_{k=\gamma}^{\infty} a_k(x-x_0)^k = \sum_{k=0}^{\infty} a_k(x-x_0)^k$$

by simply setting

$$a_k = 0 \qquad \text{when} \quad k < \gamma \quad .$$

Also, in practice, many power series are centered at 0. And even if one isn't, we can convert it to one centered at 0 via a simple change of variables:

$$\sum_{k=\gamma}^{\infty} a_k(x-x_0)^k = \sum_{k=\gamma}^{\infty} a_k X^k \qquad \text{with} \quad X = x - x_0 \quad .$$

It is also worth noting that the same sort of computations leading to equation (30.1) also yield

$$\sum_{k=\gamma}^{\infty} a_k(x-x_0)^k = (x-x_0)^{\gamma} \sum_{k=0}^{\infty} a_{\gamma+k}(x-x_0)^k \quad . \tag{30.3}$$

The basic geometric series $\sum_{k=0}^{\infty} x^k$ is a power series. From theorem 30.1, we know

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k \qquad \text{for} \quad |x| < 1 \quad . \tag{30.4a}$$

Thus, the function $(1-x)^{-1}$ can be represented by the above power series when $|x| < 1$. You may recall that many other functions can be represented by power series. Here are a few:

$$e^x = \sum_{k=0}^{\infty} \frac{1}{k!} x^k \qquad \text{for} \quad -\infty < x < \infty \quad , \tag{30.4b}$$

$$\cos(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} \qquad \text{for} \quad -\infty < x < \infty \quad , \tag{30.4c}$$

$$\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} \qquad \text{for} \quad -\infty < x < \infty \quad , \tag{30.4d}$$

and

$$\ln|x| = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} (x-1)^k \qquad \text{for} \quad 0 < x < 2 \quad . \tag{30.4e}$$

Any function that can be represented on an open interval by a power series at a point x_0 in that interval is said to be *analytic* at x_0. It turns out that many functions of interest are analytic at most points in their domains. This fact will be vital in the next several chapters.

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[3] Again, there is a minor issue with the 0^{th}-order term appearing to be 'indeterminant' when $x = x_0$. But since

$$a_0(x-x_0)^0 = a_0 \qquad \text{whenever} \quad x \neq x_0 \quad ,$$

we will, for consistency, automatically interpret $a_0(x-x_0)^0$ as a_0 even when $x = x_0$.

Convergence and the Radius of Convergence

If we are going to use a power series $\sum_{k=\gamma}^{\infty} a_k (x - x_0)^k$ as the formula for a function, it will be important to know the values of x for which this series is "makes sense" (i.e., is convergent). This set of x-values turns out to be an interval with x_0 as the midpoint. To see this, let us consider the series

$$\sum_{k=\gamma}^{\infty} a_k X^k \quad .$$

First of all, we clearly have convergence if $X = 0$ since every term with $k > 0$ is $a_k 0^k = 0$.

Now suppose we know $\sum_{k=\gamma}^{\infty} a_k r^k$ converges for some nonzero value r, and let X be any real value with $|X| < |r|$. Then $\left|{}^{X}\!/_{r}\right| < 1$ and, as noted in theorem 30.3, the geometric series

$$\sum_{k=0}^{\infty} \left|\frac{X}{r}\right|^k$$

converges. Moreover, since $\sum_{k=\gamma}^{\infty} a_k r^k$ converges, we must have $\left|a_k r^k\right| \to 0$ as $k \to \infty$, which means there must be an integer K such that $\left|a_k r^k\right| < 1$ whenever $k > K$. And that means

$$\left|a_k X^k\right| = \left|a_k r^k \frac{X^k}{r^k}\right| = \left|a_k r^k\right| \cdot \left|\frac{X}{r}\right|^k < \left|\frac{X}{r}\right|^k \qquad \text{for} \quad k > K \quad .$$

It then follows from the comparison test (theorem 30.2) using the above convergent geometric series that $\sum_{k=\gamma}^{\infty} \left|a_k X^k\right|$ converges for this choice of X. In other words,

$$|X| < |r| \quad \text{and} \quad \sum_{k=\gamma}^{\infty} a_k r^k \text{ converges} \quad \Longrightarrow \quad \sum_{k=\gamma}^{\infty} a_k X^k \text{ converges absolutely} \quad .$$

On the other hand,

$$0 < |\rho| < |X| \quad \text{and} \quad \sum_{k=\gamma}^{\infty} a_k \rho^k \text{ diverges} \quad \Longrightarrow \quad \sum_{k=\gamma}^{\infty} a_k X^k \text{ diverges} \quad ,$$

because if $\sum_{k=\gamma}^{\infty} a_k X^k$ did not diverge, then the very arguments just used in the previous paragraph would falsely imply that $\sum_{k=\gamma}^{\infty} a_k \rho^k$ converges.

Letting $X = x - x_0$, and taking r as large as possible and/or ρ as small as possible then gives the existence of the value R (which may be 0 or $+\infty$) in the next theorem.

Theorem 30.3
For each power series $\sum_{k=\gamma}^{\infty} a_k (x - x_0)^k$, there is a R — which is either 0, a finite positive value or $+\infty$ — such that

$$|x - x_0| < R \quad \Longrightarrow \quad \sum_{k=\gamma}^{\infty} a_k (x - x_0)^k \text{ converges absolutely} \quad ,$$

while

$$R < |x - x_0| \quad \Longrightarrow \quad \sum_{k=\gamma}^{\infty} a_k (x - x_0)^k \text{ diverges} \quad .$$

The R in the above theorem is called the *radius of convergence* for the given power series. If $R = 0$, the power series only converges for $x = x_0$ (which means the series won't be of much use); if $R = +\infty$, the power series converges for all values of x (which is very nice). Otherwise, the series converges absolutely at every point in the interval $(x_0 - R, x_0 + R)$. Whether we have convergence when $x = x_0 \pm R$ depends on the particular series, and, frankly, will usually not be of great concern to us.

The radius of convergence for a given power series can sometimes be determined through careful use of the formulas in either the limit ratio test or the limit root test. You may recall doing this. We, however, will discover that the radii of convergence for the power series of interest to us can be determined much more easily from the "singularities" of whatever differential equation we will be trying to solve.

Algebra with Power Series and Analytic Functions

Addition

Adding two power series with the same center and starting index is trivial:

$$\sum_{k=\gamma}^{\infty} a_k (x - x_0)^k + \sum_{k=\gamma}^{\infty} b_k (x - x_0)^k = \sum_{k=\gamma}^{\infty} \left[a_k (x - x_0)^k + b_k (x - x_0)^k \right]$$

$$= \sum_{k=\gamma}^{\infty} [a_k + b_k] (x - x_0)^k \quad .$$

However, if (as will often happen in the next chapter) they have different starting indices,

$$\sum_{k=\gamma}^{\infty} a_k (x - x_0)^k + \sum_{k=\mu}^{\infty} b_k (x - x_0)^k \qquad \text{with} \quad \gamma \neq \mu \quad ,$$

then we will first convert the series with extra low-order terms to a finite sum with those extra terms added to an infinite series with the same starting index as the other.

!►*Example 30.2:* *Consider*

$$\sum_{k=0}^{\infty} a_k X^k + \sum_{k=2}^{\infty} b_k X^k \quad .$$

Now,

$$\sum_{k=0}^{\infty} a_k X^k = a_0 + a_1 X + a_2 X^2 + a_3 X^3 + a_4 X^4 + \cdots$$

$$= a_0 + a_1 X + \sum_{k=2}^{\infty} a_k X^k \quad .$$

So,

$$\begin{aligned}\sum_{k=0}^{\infty} a_k X^k \;+\; \sum_{k=2}^{\infty} b_k X^k &= \left[a_0 \;+\; a_1 X \;+\; \sum_{k=2}^{\infty} a_k X^k\right] \;+\; \sum_{k=2}^{\infty} b_k X^k \\ &= a_0 \;+\; a_1 X \;+\; \left[\sum_{k=2}^{\infty} a_k X^k \;+\; \sum_{k=2}^{\infty} b_k X^k\right] \\ &= a_0 \;+\; a_1 X \;+\; \sum_{k=2}^{\infty} [a_k + b_k] X^k \quad .\end{aligned}$$

Changing the Index

In the next few chapters, we will often find ourselves with expressions of the form

$$\sum_{k=\gamma}^{\infty} a_k X^{k+\omega}$$

where ω is some fixed integer. On those occasions, we will want to convert this summation formula involving $X^{k+\omega}$ to an equivalent formula involving X^n. We will do this using the index substitution $n = k + \omega$ (equivalently, $k = n - \omega$),

$$\sum_{k=\gamma}^{\infty} a_k X^{k+\omega} \;=\; \sum_{n-\omega=\gamma}^{\infty} a_{n-\omega} X^n \;=\; \sum_{n=\gamma+\omega}^{\infty} a_{n-\omega} X^n \quad .$$

The goal is to end up with a power series in which each term is a constant times X^n.

This sort of index manipulation is called a *change of index*, and is analogous to the "change of variables" often used to simplify integrals. Keep in mind that the index is an "internal variable" for each series. This means we can use different index substitutions on different summations.

!►*Example 30.3:* *Consider the sum of summations*

$$\sum_{k=0}^{\infty} (k+1) a_k X^{k+2} \;+\; \sum_{k=0}^{\infty} a_k X^k \quad .$$

Using $n = k + 2$ *(i.e.,* $k = n - 2$ *) in the first summation,*

$$\sum_{k=0}^{\infty} (k+1) a_k X^{k+2} \;=\; \sum_{n-2=0}^{\infty} ([n-2]+1) a_{n-2} X^n \;=\; \sum_{n=2}^{\infty} (n-1) a_{n-2} X^n \quad .$$

For the second, we use $n = k$ *and pull out the first two terms,*

$$\sum_{k=0}^{\infty} a_k X^k \;=\; \sum_{n=0}^{\infty} a_n X^n \;=\; a_0 X^0 \;+\; a_1 X^1 \;+\; \sum_{n=2}^{\infty} a_n X^n \quad .$$

Thus,

$$\sum_{k=0}^{\infty}(k+1)a_k X^{k+2} \;+\; \sum_{k=0}^{\infty} a_k X^k \;=\; \sum_{n=2}^{\infty}(n-1)a_{n-2}X^n \;+\; \left[a_0 \;+\; a_1 X \;+\; \sum_{n=2}^{\infty} a_n X^n\right]$$

$$= \; a_0 \;+\; a_1 X \;+\; \left[\sum_{n=2}^{\infty}(n-1)a_{n-2}X^n \;+\; \sum_{n=2}^{\infty} a_n X^n\right]$$

$$= \; a_0 \;+\; a_1 X \;+\; \sum_{n=2}^{\infty}\left[(n-1)a_{n-2} + a_n\right] X^n \quad .$$

A Basic Equation

We will often find ourselves with the equation

$$\sum_{k=0}^{\infty} a_k (x-x_0)^k \;=\; 0 \qquad \text{for} \quad |x-x_0| < R \quad ,$$

which, in more explicit form (with $X = x - x_0$), is

$$a_0 \;+\; a_1 X \;+\; a_2 X^2 \;+\; a_3 X^3 \;+\; \cdots \;=\; 0 \qquad \text{for} \quad -R < X < R \quad .$$

Plugging in $X = 0$ gives

$$a_0 \;+\; \underbrace{a_1 0 \;+\; a_2 0^2 \;+\; a_3 0^3 \;+\; \cdots}_{0} \;=\; 0 \quad .$$

Hence,

$$a_0 \;=\; 0 \quad ,$$

and, for $|X| < R$,

$$a_0 \;+\; a_1 X \;+\; a_2 X^2 \;+\; a_3 X^3 \;+\; \cdots \;=\; 0$$

$$\hookrightarrow \qquad 0 \;+\; a_1 X \;+\; a_2 X^2 \;+\; a_3 X^3 \;+\; \cdots \;=\; 0$$

$$\hookrightarrow \qquad X\left(a_1 \;+\; a_2 X \;+\; a_3 X^2 \;+\; \cdots\right) \;=\; 0 \quad .$$

Assuming $R > 0$, the X factor can be divided out, leaving us with

$$a_1 \;+\; a_2 X \;+\; a_3 X^2 \;+\; \cdots \;=\; 0 \qquad \text{whenever} \quad -R < X < R \quad .$$

Plugging $X = 0$ into this last equation then gives

$$a_1 \;=\; 0 \quad .$$

Continuing this process, we can show all the a_k's are 0, thus confirming the following:

Theorem 30.4
Let $\sum_{k=0}^{\infty} a_k (x-x_0)^k$ be a power series with a nonzero radius of convergence R. If

$$\sum_{k=0}^{\infty} a_k (x-x_0)^k \;=\; 0 \qquad \textit{for} \quad |x-x_0| < R \quad ,$$

then

$$a_k = 0 \qquad \textit{for} \quad k = 0, 1, 2, 3, \ldots \quad .$$

This simple theorem will be of fundamental importance for us.

By the way, we will refer to any power series $\sum_{k=0}^{\infty} a_k(x - x_0)^k$ as a *trivial power series* if and only if all the a_k's are zero. Along the same lines, we will say that a function f analytic at a point x_0 is *trivial* if and only if it is given by a trivial power series about x_0. An immediate corollary of the above is the following unsurprising result.

Corollary 30.5

Let f be a function analytic at x_0. Then f is trivial if and only if there is an open interval (a, b) containing x_0 such that

$$f(x) = 0 \qquad \textit{whenever} \quad a < x < b \quad .$$

Naturally, our main interest will be with nontrivial analytic functions; that is, analytic functions that are not trivial.

Calculus with Power Series and Analytic Functions

Differentiating Power Series

Suppose we have a function f given by some power series with a nonzero radius of convergence R,

$$f(x) = \sum_{k=0}^{\infty} a_k(x - x_0)^k \qquad \text{for} \quad |x - x_0| < R \quad .$$

To differentiate this, it seems reasonable to use the

$$\textit{derivative of a sum} = \textit{sum of the derivatives}$$

rule from elementary calculus:

$$\begin{aligned}
f'(x) &= \frac{d}{dx} \sum_{k=0}^{\infty} a_k(x - x_0)^k \\
&= \frac{d}{dx}\big[a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + a_3(x - x_0)^3 + \cdots \\
&\qquad\qquad + a_k(x - x_0)^k + \cdots\big] \\
&= \frac{d}{dx}a_0 + \frac{d}{dx}a_1(x - x_0) + \frac{d}{dx}a_2(x - x_0)^2 + \frac{d}{dx}a_3(x - x_0)^3 + \cdots \\
&\qquad + \frac{d}{dx}a_k(x - x_0)^k + \cdots \\
&= 0 + a_1 + 2a_2(x - x_0) + 3a_3(x - x_0)^2 + \cdots + ka_k(x - x_0)^{k-1} + \cdots \\
&= \sum_{k=1}^{\infty} k\, a_k(x - x_0)^{k-1} \quad .
\end{aligned}$$

Note that the derivative of the "$k = 0$ term for $f(x)$" is 0. That is why, in the last series above, we dropped the $k = 0$ term and started with $k = 1$. Strictly speaking, this is not necessary. Since

$$ka_k(x - x_0)^{k-1} = 0 \qquad \text{when} \quad k = 0 \quad ,$$

the above series formula for f' would still be valid if it started at $k = 0$ instead of $k = 1$. Still, dropping the $k = 0$ term in the above can help prevent some embarrassing mistakes in the sort of computations we'll be doing in the next chapter.

Repeating the above (in abbreviated form) with the series obtained for $f'(x)$, we get

$$\begin{aligned} f''(x) &= \frac{d}{dx}\sum_{k=1}^{\infty} k\,a_k(x - x_0)^{k-1} \\ &= \sum_{k=1}^{\infty}\frac{d}{dx}\left[k\,a_k(x - x_0)^{k-1}\right] = \sum_{k=2}^{\infty} k(k-1)\,a_k(x - x_0)^{k-2} \quad . \end{aligned}$$

Using this, we then have

$$\begin{aligned} f'''(x) &= \frac{d}{dx}\sum_{k=2}^{\infty} k(k-1)\,a_k(x - x_0)^{k-2} \\ &= \sum_{k=2}^{\infty}\frac{d}{dx}\left[k(k-1)\,a_k(x - x_0)^{k-2}\right] = \sum_{k=3}^{\infty} k(k-1)(k-2)\,a_k(x - x_0)^{k-3} \quad . \end{aligned}$$

Continuing these computations, you end up getting

$$f^{(n)}(x) = \sum_{k=n}^{\infty} k(k-1)(k-2)\cdots(k-n+1)\,a_k(x - x_0)^{k-n}$$

for any nonnegative integer n.

There is a technical issue with the above computations. The

$$\textit{derivative of a sum} = \textit{sum of the derivatives}$$

rule from elementary calculus was only shown to be true when the sum had finitely many terms. Here we have infinitely many terms. In fact, there are infinite series of functions for which this rule fails. Fortunately, it does not fail for power series, and the following theorem can be rigorously confirmed (see any good calculus text).

Theorem 30.6 (differentiation of power series)

Suppose f is a function given by a power series with a nonzero radius of convergence R,

$$f(x) = \sum_{k=0}^{\infty} a_k(x - x_0)^k \qquad \text{for} \quad |x - x_0| < R \quad .$$

Then, for any positive integer n, the n^{th} derivative of f exists. Moreover, R is also the radius of convergence of the differentiated series, with

$$f^{(n)}(x) = \sum_{k=n}^{\infty} k(k-1)(k-2)\cdots(k-n+1)\,a_k(x - x_0)^{k-n} \qquad \text{for} \quad |x - x_0| < R \quad .$$

In particular,

$$f'(x) = \sum_{k=1}^{\infty} k\, a_k (x - x_0)^{k-1} \qquad \text{for} \quad |x - x_0| < R \quad ,$$

and

$$f''(x) = \sum_{k=2}^{\infty} k(k-1)\, a_k (x - x_0)^{k-2} \qquad \text{for} \quad |x - x_0| < R \quad .$$

Integral analogs to the above theorem also hold. In particular, if

$$f(x) = \sum_{k=0}^{\infty} a_k (x - x_0)^k \qquad \text{for} \quad |x - x_0| < R \quad ,$$

then it can be verified that

$$\int_{x_0}^{x} f(t)\, dt = \sum_{k=0}^{\infty} \int_{x_0}^{x} a_k (t - x_0)^k\, dt = \sum_{k=0}^{\infty} \frac{a_k}{k+1} (x - x_0)^{k+1}$$

whenever $|x - x_0| < R$. This can be a useful fact, though we won't have much need for it.

Power Series for Analytic Functions

As already noted, any function f given by a power series centered at x_0 in some open interval containing x_0 is said to be *analytic* at x_0 . If, in addition, f is analytic at every point in some interval, then we say f is *analytic on that interval*.

So suppose we have a function f analytic at x_0 with

$$f(x) = \sum_{k=0}^{\infty} a_k (x - x_0)^k \qquad \text{for} \quad |x - x_0| < R$$

for some $R > 0$. Our 'differentiation of power series' theorem (theorem 30.6) tells us that f is, in fact, "infinitely differentiable" on the interval $(x_0 - R, x_0 + R)$.[4] That theorem also allows us to derive a simple relationship between the a_k's in the series and the derivatives of f at x_0 .

Let's derive that relation: First, plugging $x = x_0$ into the above, we get

$$\begin{aligned} f(x_0) &= \sum_{k=0}^{\infty} a_k (x_0 - x_0)^k \\ &= a_0 + a_1(x_0 - x_0) + a_2(x_0 - x_0)^2 + a_3(x_0 - x_0)^3 + \cdots \\ &= a_0 + a_1 \cdot 0 + a_2 \cdot 0^2 + a_3 \cdot 0^3 + \cdots \\ &= a_0 \quad . \end{aligned}$$

Then, using formulas from theorem 30.6, we see that

$$\begin{aligned} f'(x_0) &= \sum_{k=1}^{\infty} k\, a_k (x_0 - x_0)^{k-1} \\ &= 1 a_1 + 2 a_2 \cdot 0 + 3 a_3 \cdot 0^2 + 4 a_4 \cdot 0^3 + \cdots \\ &= 1 a_1 \quad , \end{aligned}$$

---

[4] We say that a function f is *infinitely differentiable* at a point x if and only if $f^{(n)}(x)$ exists for every positive integer n , and is infinitely differentiable on a given interval if and only if it is infinitely differentiable at each point in the interval.

and

$$\begin{aligned} f''(x_0) &= \sum_{k=2}^{\infty} k(k-1)\, a_k (x_0 - x_0)^{k-2} \\ &= 2 \cdot 1\, a_2 + 3 \cdot 2\, a_3 \cdot 0 + 4 \cdot 3\, a_4 \cdot 0^2 + 5 \cdot 4\, a_5 \cdot 0^3 + \cdots \\ &= (2 \cdot 1) a_2 \quad . \end{aligned}$$

More generally, for any positive integer n,

$$\begin{aligned} f^{(n)}(x_0) &= \sum_{k=n}^{\infty} k(k-1)(k-2)\cdots(k-n+1)\, a_k (x_0 - x_0)^{k-n} \\ &= n(n-1)(n-2)\cdots 1\, a_n \\ &\quad + (n+1)(n)(n-1)\cdots 2\, a_{n+1} \cdot 0 + (n+2)(n+1)(n)\cdots 3\, a_{n+2} \cdot 0^2 \\ &\quad + \cdots \\ &= n!\, a_n \quad . \end{aligned}$$

Dividing the last relation through by $n!$ and observing that the result also holds when $n = 0$ (interpreting $f^{(0)}$ as f) gives us the next theorem.

Theorem 30.7
Let f be analytic at x_0. Then, for every x in some open interval containing x_0,

$$f(x) = \sum_{k=0}^{\infty} a_k (x - x_0)^k \qquad \text{with} \quad a_k = \frac{f^{(k)}(x_0)}{k!} \quad .$$

As an immediate corollary, we have the following (which will be important when discussing "power series solutions to initial-value problems"):

Corollary 30.8
Assume

$$f(x) = \sum_{k=0}^{\infty} a_k (x - x_0)^k \qquad \text{for} \quad |x - x_0| < R$$

with $R > 0$. Then,

$$a_0 = f(x_0) \qquad \text{and} \qquad a_1 = f'(x_0) \quad .$$

You may recognize the series in theorem 30.7, written a little more simply as

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k \quad ,$$

as the *Taylor's series (formula)* for $f(x)$ about x_0, and you may recall having once computed Taylor series for such functions as e^x, $\sin(x)$, $\cos(x)$ and $\ln|x|$. In fact, the Taylor series about any point x_0 can be computed for any function f which is infinitely differentiable at that point. However, a function can be infinitely differentiable at a point x_0 without being analytic there — its Taylor series exists, but does not equal the function at any point other than x_0. With luck you saw an example, say,

$$f(x) = \begin{cases} 0 & \text{if} \quad x = 0 \\ e^{-1/x^2} & \text{if} \quad x \neq 0 \end{cases} ,$$

which can be shown to be infinitely differentiable but not analytic at $x_0 = 0$ (see exercise 30.10).

Still, you probably saw that many functions are analytic at many points. You may well have already verified that such functions as

$$e^x \quad , \quad e^{-2x^2} \quad , \quad \sin(x) \quad \text{and} \quad \cos(x)$$

are analytic at every point on the real line, and that functions such as

$$\sqrt{x} \quad \text{and} \quad \ln x$$

are analytic at any point $x_0 > 0$. You may have even had been given the impression that most functions typically encountered in "real life" are analytic at every point at which they are infinitely differentiable. In a sense, this is true, though very difficult to confirm using the methods normally developed in elementary calculus courses. (We'll return to this issue in chapter 32.)

30.3 Elementary Complex Analysis

Up to now, we've acted as if we were only dealing with real numbers in our infinite series. In fact, just about everything said so far, up to the discussion of *Calculus with Power Series*, holds even if the numbers are complex, provided we make some obvious changes in notation and phrasing.[5] In fact, we will later have particular interest in power series in which the variables are complex.

The Complex Plane

Recall that a *complex number* z is simply something that can be written as

$$z = x + iy$$

where x and y are real numbers, and i is a constant satisfying $i^2 = -1$. Because we'll be using such expressions so often, let us agree that, unless otherwise noted, in any expression of the form $z = x + iy$, both x and y are real numbers.

As you are probably aware, each complex number $z = x + iy$ can be identified with the point (x, y) in the XY–plane. When we do so, we generally refer to the plane as the *complex plane* and denote it by $\mathbb{C}$. The distance between any two points

$$z_1 = x_1 + iy_1 \quad \text{and} \quad z_2 = x_2 + iy_2$$

in the complex plane is simply their distance as points in the XY–plane,

$$|z_2 - z_1| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \quad .$$

[5] And, after discussing "complex calculus" in chapter 32, we'll discover that what was said in *Calculus with Power Series* also holds for power series with a complex variable.

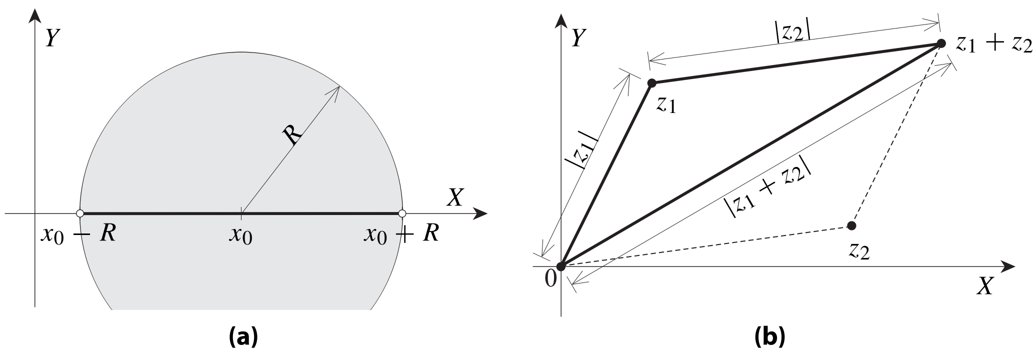

Figure 30.1: (a) The disk of radius R about x_0 in $\mathbb{C}$ with the interval $(x_0 - R, x_0 + R)$ on the X–axis, and **(b)** an illustration of the triangle inequality.

Note that, for any z_0 in $\mathbb{C}$ and $R > 0$, the set of all z satisfying $|z - z_0| < R$ is the disk of radius R centered at z_0. For comparison, recall that, when we were just considering real numbers, the set of x satisfying $|x - x_0| < R$ was the interval $(x_0 - R, x_0 + R)$ (see figure 30.1a).

Power Series and Analytic Functions

If you look at the triangle having vertices 0, z_1, z_2 and $z_1 + z_2$ (as in figure 30.1b), you'll realize that the triangle inequality,

$$|z_1 + z_2| \leq |z_1| + |z_2| \quad ,$$

holds for complex as well as real values. Using this fact, everything already stated regarding "absolute convergence" is easily shown to apply whether a series involves real or complex values. In particular, theorem 30.3 on page 30–8 can be automatically expanded to

Theorem 30.9
For each power series $\sum_{k=\gamma}^{\infty} a_k(z - z_0)^k$, there is a R — which is either 0, a finite positive value or $+\infty$ — such that

$$|z - z_0| < R \quad \Longrightarrow \quad \sum_{k=\gamma}^{\infty} a_k(z - z_0)^k \text{ converges absolutely} \quad ,$$

while

$$R < |z - z_0| \quad \Longrightarrow \quad \sum_{k=\gamma}^{\infty} a_k(z - z_0)^k \text{ diverges} \quad .$$

As before, we call the R in this theorem the *radius of convergence* for the power series, and we can refer to the point z_0 as the *center* for our series. This time, the terminology is truly appropriate, since the given power series does converge inside the disk of radius R about z_0, and diverges outside that disk.

Along these same lines, we extend the definition of analyticity to functions of complex variables by saying that a function f of a complex variable z is *analytic at a point z_0 in the* *complex plane* if and only if there is a power series $\sum_{k=0}^{\infty} a_k(z-z_0)^k$ and a $R > 0$ (possibly with $R = \infty$), such that

$$f(z) = \sum_{k=0}^{\infty} a_k(z-z_0)^k \qquad \text{for} \quad |z-z_0| < R \quad .$$

Naturally, we say that the function is analytic on a region in the complex plane if and only if it is analytic at each point in that plane.[6]

Let us note that, if f is a function of a real variable x given by, say,

$$f(x) = \sum_{k=0}^{\infty} a_k(x-x_0)^k \qquad \text{for} \quad |x-x_0| < R \quad ,$$

then f can be extended to a function of the complex variable by simply replacing x in the series with z,

$$f(z) = \sum_{k=0}^{\infty} a_k(z-x_0)^k \qquad \text{whenever} \quad |z-x_0| < R \quad .$$

What is more, it follows directly from theorem 30.9 that the radii of convergence for both of the two series above are the same. In particular, the corresponding complex variable versions of the functions given by formulas 30.4 on page 30–7 are

$$\frac{1}{1-z} = \sum_{k=0}^{\infty} z^k \qquad \text{for} \quad |z| < 1 \quad , \tag{30.5a}$$

$$e^z = \sum_{k=0}^{\infty} \frac{1}{k!} z^k \qquad \text{for} \quad |z| < \infty \quad , \tag{30.5b}$$

$$\cos(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} z^{2k} \qquad \text{for} \quad |z| < \infty \quad , \tag{30.5c}$$

$$\sin(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} z^{2k+1} \qquad \text{for} \quad |z| < \infty \quad , \tag{30.5d}$$

and

$$\ln|z| = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} (z-1)^k \qquad \text{for} \quad |z-1| < 1 \quad . \tag{30.5e}$$

There is a issue here that may concern the thoughtful reader: Some of the above functions have formulas other than the above power series for computing their values at complex points. For example, in chapter 16, we learned of another formula for e^z when $z = x + iy$; namely,

$$e^z = e^{x+iy} = e^x \left[\cos(y) + i\sin(y)\right] \quad .$$

Can we be sure that this formula will give the same result as using the above power series for e^z,

$$\sum_{k=0}^{\infty} \frac{1}{k!} z^k = \sum_{k=0}^{\infty} \frac{1}{k!} (x+iy)^k \quad ?$$

Yes, we can. Trust the author on this. And if you don't feel that trust, turn ahead to section 32.7 (starting on page 32–18) where we discuss the calculus of functions of a complex variable.

[6] If you've had a course in complex analysis, you may have seen a different definition for "analyticity". In section 32.7, we will find that the two definitions are equivalent.

30.4 Additional Basic Material That May Be Useful

The material in the previous sections will be needed in the next chapter. But there are some additional facts about series and power series that will be useful later, especially when we get deeper into the rigorous theory behind the computations that we will be developing. For convenience, we'll provide some the more basic general facts here, and develop the more advanced material as needed. It won't hurt to skip this material initially, provided you return to it as needed.

Two More General Tests for Convergence

The well-known basic comparison test for determining if a given series converges or diverges was described in theorem 30.2 on page 30–6 and was used in developing the radius of convergence for power series. In chapter 32, we will find the next test, a clever refinement of the basic comparison test, to be useful.

Theorem 30.10 (the limit comparison test)
Let $\sum_{k=\gamma}^{\infty} \alpha_k$ and $\sum_{k=\mu}^{\infty} \beta_k$ be two infinite series, and suppose

$$\lim_{k\to\infty} \left| \frac{\alpha_k}{\beta_k} \right|$$

exists as either a finite number or as $+\infty$. Then

$$\lim_{k\to\infty} \left| \frac{\alpha_k}{\beta_k} \right| < \infty \quad \text{and} \quad \sum_{k=\mu}^{\infty} |\beta_k| \text{ converges} \quad \Longrightarrow \quad \sum_{k=\gamma}^{\infty} |\alpha_k| \text{ converges} \quad ,$$

while

$$\lim_{k\to\infty} \left| \frac{\alpha_k}{\beta_k} \right| > 0 \quad \text{and} \quad \sum_{k=\mu}^{\infty} |\beta_k| \text{ diverges} \quad \Longrightarrow \quad \sum_{k=\gamma}^{\infty} |\alpha_k| \text{ diverges} \quad .$$

Under certain conditions, you can use the limit of the ratio of the consecutive terms of a single series to construct a geometric series that can serve as a second series in the above limit comparison test. That leads to a third test, which will be used near the end of chapter 35.

Theorem 30.11 (the limit ratio test)
Let $\sum_{k=\gamma}^{\infty} \alpha_k$ be an infinite series, and suppose

$$\lim_{k\to\infty} \left| \frac{\alpha_{k+1}}{\alpha_k} \right|$$

exists as either a finite number or as $+\infty$. Then

$$\lim_{k\to\infty} \left| \frac{\alpha_{k+1}}{\alpha_k} \right| < 1 \quad \Longrightarrow \quad \sum_{k=\gamma}^{\infty} \alpha_k \text{ converges absolutely} \quad ,$$

while

$$\lim_{k\to\infty} \left| \frac{\alpha_{k+1}}{\alpha_k} \right| > 1 \quad \Longrightarrow \quad \sum_{k=\gamma}^{\infty} \alpha_k \text{ diverges} \quad .$$

(If the limit is 1 *, there is no conclusion.)*

The derivations of the above tests can be found in any reasonable elementary calculus text.

More on Algebra with Power Series and Analytic Functions

Multiplication

The following — a straightforward extension of a basic formula for computing products of polynomials — is worth a brief mention, especially since we will need it in chapter 32.

Theorem 30.12

The product of two power series centered at the same point is another power series whose radius of convergence is at least as large as the smallest radius of convergence of the original two series. Moreover,

$$\left(\sum_{k=0}^{\infty} a_k(z-z_0)^k\right)\left(\sum_{k=0}^{\infty} b_k(z-z_0)^k\right) = \sum_{k=0}^{\infty} c_k(z-z_0)^k$$

with

$$c_k = a_0b_k + a_1b_{k-1} + \cdots + a_kb_0 = \sum_{j=0}^{k} a_j b_{n-j} \quad .$$

Factoring a Power Series/Analytic Function

On occasion, it will be convenient to "factor out" factors of the form $(z-z_0)^m$ from a function f analytic at z_0. Our ability to do this follows immediately from the fact that, being analytic at z_0, f is given by some power series with a nonzero radius of convergence R,

$$f(z) = \sum_{k=0}^{\infty} a_k(z-z_0)^k \qquad \text{for} \quad |z-z_0| < R \quad .$$

Note that

$$f(z_0) = a_0 + a_1(z_0-z_0) + a_2(z_0-z_0)^2 + \cdots = a_0 \quad ,$$

telling us that $f(z_0) = 0$ if and only if $a_0 = 0$.

Let's go a little further and assume $f(z_0) = 0$, Then, as just noted, $a_0 = 0$. Moreover,

$$\begin{aligned} f(z) &= a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + a_3(z-z_0)^3 + \cdots \\ &= 0 + (z-z_0)\left[a_1 + a_2(z-z_0) + a_3(z-z_0)^2 + \cdots\right] \\ &= (z-z_0)\sum_{k=1}^{\infty} a_k(z-z_0)^k \qquad \text{for} \quad |z-z_0| < R \quad . \end{aligned}$$

Of course, we could also have $a_1 = 0$, in which case we can repeat the above to obtain

$$f(z) = (z-z_0)^2\sum_{k=2}^{\infty} a_k(z-z_0)^k \qquad \text{for} \quad |z-z_0| < R \quad .$$

Continuing until we finally reach a nonzero coefficient (assuming f in nontrivial), we get, for some positive integer m ,

$$f(z) = (z - z_0)^m \sum_{k=m}^{\infty} a_k (z - z_0)^k \qquad \text{with} \quad a_m \neq 0 \quad .$$

Then letting

$$f_0(z) = \sum_{k=m}^{\infty} a_k (z - z_0)^k$$

and noting that, even if $a_0 \neq 0$, it is trivially true that

$$f(z) = (z - z_0)^0 f(z) \quad ,$$

we get

Lemma 30.13
Let f be a nontrivial function analytic at z_0 . Then there is a nonnegative integer m such that

$$f(z) = (z - z_0)^m f_0(z_0)$$

where f_0 is a function analytic at z_0 with $f(z_0) \neq 0$. Moreover:

1. $f(z_0) = 0$ *if and only if* $m > 0$.
2. *The power series for f and f_0 about z_0 have the same radii of convergence.*

It is standard to refer to any point z_0 as a *zero* for an analytic function if $f(z_0) = 0$. It is also standard to refer to the m described in the last lemma as the *multiplicity* of the zero z_0 .

Quotients of Analytic Functions

There is a technical issue that may arise when defining a function h as the quotient of two functions analytic at a given point.

!►*Example 30.4:* *Consider defining $h = {}^f/_g$ when f and g are the polynomials*

$$f(z) = z^2 - 1 \qquad \textit{and} \qquad g(z) = z - 1 \quad .$$

For any value of z other than $z = 1$, we simply have

$$h(z) = \frac{f(z)}{g(z)} = \frac{z^2 - 1}{z - 1} \quad ,$$

which we can rewrite more simply by dividing out the common factor,

$$h(z) = \frac{z^2 - 1}{z - 1} = \frac{(z - 1)(z + 1)}{z - 1} = z + 1 \quad .$$

These two formulas for $h(z)$ give identical results whenever $z \neq 1$. However, we get two different results if we plug in $z = 1$:

$$h(1) = \frac{1^2 - 1}{1 - 1} = \frac{0}{0} \qquad \textit{and} \qquad h(1) = 1 + 1 = 2 \quad .$$

The first expression is indeterminant, and is problematic in practical applications. The second is an finite number, and is clearly what we want to use for $h(1)$, especially since:

1. *It comes from a simpler formula for $h(z)$ when $z \neq 1$.*

2. *It is the same as the value we would obtain using the obvious limit,*

$$\lim_{z\to 1} h(z) = \lim_{z\to 1} h(z) = \lim_{z\to 1} \frac{f(z)}{g(z)} = \lim_{z\to 1} \frac{z^2-1}{z-1} = \cdots = 2 \quad ,$$

computed either using L'Hôpital's rule or the formula obtained by dividing out the common factors. Hence, h is continuous at $z = 1$.

More generally, when we define a function h as the quotient of two other continuous functions, $h = {}^{f}/_{g}$, we automatically mean the function given by

$$h(z_0) = \frac{f(z_0)}{g(z_0)}$$

at each point z_0 in the common domain of f and g at which $g(z_0) \neq 0$; and, provided the limit exists as a finite number, by

$$h(z_0) = \lim_{z\to z_0} \frac{f(z)}{g(z)}$$

at each point z_0 in the common domain of f and g at which $g(z_0) = 0$. (If the above limit does not exist, we simply accept that h is not defined at that z_0 .)

We should note that using the limit when f and g are analytic and zero at z_0 yields exactly the same as if we were to divide out any common factors of $z - z_0$ in the quotient. After all, if f and g are analytic, then (as shown in the previous subsection) we can rewrite $f(z)$ and $g(z)$ as

$$f(z) = (z-z_0)^m f_0(z) \qquad \text{and} \qquad g(z) = (z-z_0)^n g_0(z)$$

where m and n are nonnegative integers, and f_0 and g_0 are functions analytic at z_0 with $f_0(z_0) \neq 0$ and $g_0(z_0) \neq 0$. Hence,

$$\frac{f(z)}{g(z)} = \frac{(z-z_0)^m f_0(z)}{(z-z_0)^n g_0(z)} = (z-z_0)^{m-n} \frac{f_0(z)}{g_0(z)} \quad , \tag{30.6}$$

and

$$\lim_{z\to z_0} \frac{f(z)}{g(z)} = \lim_{z\to z_0} (z-z_0)^{m-n} \frac{f_0(z_0)}{g_0(z_0)} \quad ,$$

which is finite if and only if $m \geq n$. Since this observation will later be used, let us make it a lemma.

Lemma 30.14

Let f and g be two functions analytic at z_0 . Then

$$\lim_{z\to z_0} \frac{f(z)}{g(z)}$$

exists and is finite if and only if there is a nonnegative integer N and a $R > 0$ such that

$$\frac{f(z)}{g(z)} = (z-z_0)^N \frac{f_0(z)}{g_0(z)}$$

where f_0 and g_0 are functions analytic at z_0 with $f_0(z_0) \neq 0$ and $g_0(z_0) \neq 0$.

Partial Sum Approximations with Taylor Series

Another approach to deriving the Taylor series formula for function f analytic at a point x_0 starts with the well-known equality

$$f(x) - f(x_0) = \int_{x_0}^{x} f'(s)\, ds \quad .$$

Solving for $f(x)$ and using a "clever" integration by parts yields

$$\begin{aligned}
f(x) &= f(x_0) + \int_{x_0}^{x} \underbrace{f'(s)}_{u} \underbrace{ds}_{dv} \\
&= f(x_0) + \left[\underbrace{f'(s)}_{u}\underbrace{(s - x_0)}_{v}\Big|_{s=x_0}^{x} - \int_{x_0}^{x} \underbrace{(s - x_0)}_{v} \underbrace{f''(s)\, ds}_{du} \right] \\
&= f(x_0) + f'(x)(x - x) - f'(x_0)(x_0 - x) - \int_{x_0}^{x} (s - x_0) f''(s)\, ds \\
&= f(x_0) + 0 + f'(x_0)(x - x_0) - \int_{x_0}^{x} (s - x_0) f''(s)\, ds \quad .
\end{aligned}$$

Repeating this again and again with similar "clever" uses of integration by parts ultimately leads to

$$f(x) = P_N(x) + E_N(x) \qquad \text{for} \quad N = 1, 2, 3, \ldots \tag{30.7a}$$

where $P_N(x)$ is the N^{th} *degree Taylor polynomial*,

$$P_N(x) = \sum_{k=0}^{N} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k \quad , \tag{30.7b}$$

and $E_N(x)$ is the corresponding remainder term,

$$E_N(x) = (-1)^N \frac{1}{N!} \int_{x_0}^{x} f^{(N+1)}(s)(s - x)^N\, ds \quad . \tag{30.7c}$$

Note that $P_N(x)$ is the N^{th} partial sum for the power series for f about x_0, and $E_N(x)$ is the error in using this partial sum in place of $f(x)$.

Since we are assuming f is analytic at x_0, we know that

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k = \lim_{N\to\infty} P_N(x) \qquad \text{whenever} \quad |x - x_0| < R \quad ,$$

where R is the radius of convergence for the Taylor series. This, in turn, means that

$$\lim_{N\to\infty} E_N(x) = \lim_{N\to\infty} [f(x) - P_N(x)] = 0 \qquad \text{whenever} \quad |x - x_0| < R \quad .$$

Conversely, in theory, you can verify that f truly is analytic at x_0 and that its power series at that point has radius of convergence of at least R by verifying that $E_N(x) \to 0$ as $N \to \infty$ whenever $x_0 - R < x < x_0 + R$.

In practice, computing $E_N(x)$ is rarely practical. Because of this, a slightly more useable error estimate in terms of upper bounds on the derivatives of f is often described in textbooks. For our purposes, let $[a, b]$ be some closed subinterval with

$$x_0 - R \;<\; a \;<\; x_0 \;<\; b \;<\; x_0 + R \quad .$$

By the analyticity of f, we know $\left|f^{(N+1)}\right|$ is continuous and, thus, has an upper bound on $[a, b]$; that is, there is a finite value M_N such that

$$\left|f^{(N+1)}(x)\right| \;\leq\; M_N \qquad \text{for} \quad a < x < b \quad .$$

Then, as you can easily verify,

$$\begin{aligned} |E_N(x)| \;&=\; \left|\frac{1}{N!}\int_{x_0}^{x} f^{(N+1)}(s)(s-x)^N\,ds\right| \\ &\leq\; \left|\frac{1}{N!}\int_{x_0}^{x} M_N(s-x)^N\,ds\right| \;=\; \frac{M_N}{(N+1)!}\,|x-x_0|^{N+1} \quad . \end{aligned}$$

The value of this estimate in showing that $\lim_{N\to\infty} E_N(x) \to 0$ depends on being able to find the appropriate upper bound M_N for each positive integer N. Unfortunately for us, finding these M_N's will not be practical. So we will not be using $E_N(x)$ to determine the analyticity of f or the radius of convergence for its power series.

So why go through this discussion of E_N? It is for this simple observation regarding a function f analytic at x_0: For any given positive integer N,

$$\frac{M_N}{(N+1)!}\,|x-x_0|^{N+1} \;\to\; 0 \quad \text{“quickly” as} \quad x \;\to\; x_0 \quad .$$

Hence, for any given positive integer N,

$$E_N(x) \;\to\; 0 \quad \text{“quickly” as} \quad x \;\to\; x_0 \quad ,$$

assuring us that the N^{th} partial sum

$$\sum_{k=0}^{N} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$$

is a very good approximation to $f(x)$ on some (possibly small) open interval about x_0.

Additional Exercises

30.2. *Several expressions involving geometric series are given below. If the given expression is a partial sum or a convergent infinite series, compute it's sum using the formulas developed for geometric series and their partial sums. If the given expression is a divergent series, say so.*

a. $\displaystyle\sum_{k=0}^{4}\left(\frac{1}{3}\right)^k$ **b.** $\displaystyle\sum_{k=0}^{8}\left(\frac{1}{3}\right)^k$ **c.** $\displaystyle\sum_{k=0}^{\infty}\left(\frac{1}{3}\right)^k$ **d.** $\displaystyle\sum_{k=5}^{\infty}\left(\frac{1}{3}\right)^k$

e. $\displaystyle\sum_{k=0}^{\infty}\left(-\frac{2}{3}\right)^k$ **f.** $\displaystyle\sum_{k=0}^{5}\left(\frac{3}{2}\right)^k$ **g.** $\displaystyle\sum_{k=0}^{\infty}\left(\frac{3}{2}\right)^k$ **h.** $\displaystyle\sum_{k=0}^{\infty}8\left(\frac{3}{7}\right)^k$

i. $\displaystyle\sum_{k=0}^{\infty}\left[3\left(\frac{2}{5}\right)^k-4\left(\frac{3}{5}\right)^k\right]$ **j.** $\displaystyle\sum_{k=0}^{\infty}\left[3\left(\frac{1}{10}\right)^k+\frac{2}{3}\left(-\frac{3}{5}\right)^k\right]$

30.3. *Verify each of the following equations:*

a. $\displaystyle\sum_{k=0}^{\infty}\frac{1}{k+1}x^k + \sum_{k=2}^{\infty}\frac{1}{k-1}x^k = 1 + \frac{1}{2}x + \sum_{k=2}^{\infty}\frac{2k}{k^2-1}x^k$

b. $\displaystyle\sum_{k=0}^{\infty}\left(k^2+9\right)x^k - 6\sum_{k=1}^{\infty}kx^k = 9 + \sum_{k=1}^{\infty}(k-3)^2x^k$

c. $\displaystyle\sum_{k=2}^{\infty}(k-1)x^{k-2} = \sum_{n=0}^{\infty}(n+1)x^n$

d. $\displaystyle x\sum_{k=2}^{\infty}(k-1)x^{k-2} = \sum_{n=1}^{\infty}nx^n$

e. $\displaystyle\sum_{k=0}^{\infty}(k+1)x^{k+1} - \sum_{k=4}^{\infty}(k-1)x^{k-1} = x+2x^2$

f. $\displaystyle x^3\sum_{k=0}^{\infty}a_kx^k = \sum_{n=3}^{\infty}a_{n-3}x^n$

g. $\displaystyle\left(x^2+5\right)\sum_{k=0}^{\infty}a_kx^k = 5a_0 + 5a_1x + \sum_{n=2}^{\infty}[a_{n-2}+5a_n]x^n$

h. $\displaystyle\sum_{k=2}^{\infty}k(k-1)a_kx^{k-2} - 3\sum_{k=0}^{\infty}a_kx^k = \sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2}-3a_n\right]x^n$

30.4. *Rewrite each of the following expressions as a single power series centered at a point x_0, with the index being the order of each term. That is, if n is the index, then each term should be of the form*

$$[\textit{formula not involving } x] \times (x-x_0)^n \quad .$$

In most cases, $x_0 = 0$. And, in some cases, the first few terms will have to be written separately. Simplify your expressions as much as practical.

a. $\displaystyle\sum_{k=0}^{\infty}\frac{1}{k+1}x^k - \sum_{k=1}^{\infty}\frac{1}{k}x^k$ **b.** $\displaystyle x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \sum_{k=1}^{\infty}\frac{(-1)^k}{k}x^k$

c. $\displaystyle\sum_{k=1}^{\infty}3k^2(x-5)^{k+3}$ **d.** $\displaystyle x\sum_{k=2}^{\infty}k(k-1)x^{k-2}$

e. $(x-3)\sum_{k=2}^{\infty} k(k-1)x^{k-2}$

f. $x\sum_{k=2}^{\infty} k(k-1)(x-3)^{k-2}$

g. $\sum_{k=1}^{\infty} k^2 a_k x^{k+3}$

h. $\sum_{k=0}^{\infty}(k+1)a_k x^{k+1} - \sum_{k=4}^{\infty}(k-1)a_k x^{k-1}$

i. $(x-1)^2\sum_{k=0}^{\infty} a_k (x-1)^k$

j. $\sum_{k=1}^{\infty} k a_k x^{k-1} + 5\sum_{k=0}^{\infty} a_k x^k$

k. $x^2\sum_{k=2}^{\infty} k(k-1)a_k x^{k-2} - 4\sum_{k=0}^{\infty} a_k x^k$

l. $\sum_{k=2}^{\infty} k(k-1)a_k x^{k-2} - 3x^2\sum_{k=0}^{\infty} a_k x^k$

30.5. *On page 30–5, we saw that*

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k \qquad \text{for} \quad |x| < 1 \quad .$$

By differentiating this, find the power series about 0 *for each of the following:*

a. $\dfrac{1}{(1-x)^2}$ **b.** $\dfrac{1}{(1-x)^3}$

30.6. *Find the Taylor series about* x_0 *for each of the following:*

a. e^x *with* $x_0 = 0$

b. $\cos(x)$ *with* $x_0 = 0$

c. $\sin(x)$ *with* $x_0 = 0$

d. $\ln|x|$ *with* $x_0 = 1$

30.7 a. *Using the Taylor series formula from theorem 30.7, find the fourth partial sum of the power series about* 0 *for*

$$f(x) = \sqrt{1+x} \quad .$$

b. *Using the results from the previous part, derivatives. algebra, index manipulation, etc., find the first four terms of the power series about* 0 *for the following:*

i. $\dfrac{1}{\sqrt{1+x}}$ **ii.** $\dfrac{1}{\sqrt{1+x^2}}$

30.8 a. *Using your favorite computer mathematics package (e.g., Maple or Mathematica), along with the Taylor series formula from theorem 30.7, write a program/worksheet that will find the first* N *coefficients in the power series about* x_0 *for* f *where* x_0 *is any given point on the real line,* f *any function analytic at* x_0 *, and* N *is any given positive integer. Also, have your program/worksheet write out the corresponding* N^{th}*-degree partial sum of this power series. Be sure to write your program/worksheet so that* N *,* x_0 *and* f *are easily changed.*

b. *Use your program/worksheet with each of the following choices of* f *,* x_0 *and* N *to find the* N^{th}*-degree polynomial about* x_0 *for* f *.*

i. $f(x) = e^{2x}$ *with* $x_0 = 0$ *and* $N = 9$

ii. $f(x) = \dfrac{1}{\cos(x)}$ *with* $x_0 = 0$ *and* $N = 11$

iii. $f(x) = \sqrt{2x^2+1}$ *with* $x_0 = 2$ *and* $N = 7$

30.9. *We saw that*

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k \qquad \text{for} \quad |x| < 1 \quad .$$

Replacing x *with* $-x$,

$$\frac{1}{1-(-x)} = \sum_{k=0}^{\infty} (-x)^k \qquad \text{for} \quad |-x| < 1 \quad ,$$

gives us a power series formula for $(1+x)^{-1}$,

$$\frac{1}{1+x} = \sum_{k=0}^{\infty} (-x)^k \qquad \text{for} \quad |-x| < 1 \quad .$$

Find a power series representation (and its radius of convergence R *) for the each of the following by replacing the* x *in some of the "known" power series from exercises 30.5 and 30.6, above, with a suitable formula of* x *, as just done above.*

a. $\dfrac{1}{1-2x}$ **b.** $\dfrac{1}{1+x^2}$ **c.** $\dfrac{2}{2-x}$ **d.** $\dfrac{2}{(2-x)^2}$

e. e^{-x^2} **f.** $\sin(x^2)$

30.10. *Let*

$$f(x) = \begin{cases} 0 & \text{if} \quad x = 0 \\ e^{-1/x^2} & \text{if} \quad x \neq 0 \end{cases} \quad .$$

a. *Verify that*

$$f'(x) = \begin{cases} 0 & \text{if} \quad x = 0 \\ e^{-1/x^2}\dfrac{2}{x^3} & \text{if} \quad x \neq 0 \end{cases}$$

Note: The derivative at $x = 0$ *should be computed using the basic definition*

$$f'(0) = \lim_{\Delta x \to 0} \frac{f(0+\Delta x) - f(0)}{\Delta x} \quad .$$

b. *Verify that*

$$f''(x) = \begin{cases} 0 & \text{if} \quad x = 0 \\ e^{-1/x^2}\dfrac{1}{x^6}\left[4 - 6x^2\right] & \text{if} \quad x \neq 0 \end{cases} \quad .$$

c. *Verify that, for any positive integer* k

$$f^{(k)}(x) = \begin{cases} 0 & \text{if} \quad x = 0 \\ e^{-1/x^2}\dfrac{1}{x^{3k}}p_k(x) & \text{if} \quad x \neq 0 \end{cases}$$

Where $p_k(x)$ *is some polynomial.*

d. *Using the above results, write out the Taylor series for* f *about* 0 .

e. *Why is* f *not analytic at* 0 *even though it is infinitely differentiable at* 0 *?*

Some Answers to Some of the Exercises

WARNING! Most of the following answers were prepared hastily and late at night. They have not been properly proofread! Errors are likely!

2a. $\frac{121}{81}$

2b. $\frac{9{,}841}{6{,}561}$

2c. $\frac{3}{2}$

2d. $\frac{1}{162}$

2e. $\frac{3}{5}$

2f. $\frac{665}{32}$

2g. Diverges

2h. 14

2i. -5

2j. $\frac{15}{4}$

4a. $1 + \sum_{k=1}^{\infty} \frac{-1}{k(k+1)} x^k$

4b. $x^2 + \sum_{k=4}^{\infty} \frac{(-1)^k}{k} x^k$

4c. $\sum_{n=4}^{\infty} 3(n-3)^2(x-5)^n$

4d. $\sum_{n=1}^{\infty} (n+1)nx^n$

4e. $-6 + \sum_{n=1}^{\infty} [-2(n+1)(n+3)]x^n$

4f. $6 + \sum_{n=1}^{\infty} [2(n+1)(2n+3)]x^n$

4g. $\sum_{n=4}^{\infty} (n-3)^2 a_{n-3} x^n$

4h. $a_0 x + 2a_1 x^2 + \sum_{n=3}^{\infty} n(a_{n-1} - a_{n+1})x^n$

4i. $\sum_{n=2}^{\infty} a_{n-2}(x-1)^n$

4j. $\sum_{n=0}^{\infty} \left[(n+1)a_{n+1} + 5a_n\right] x^n$

4k. $-4a_0 - 4a_1 x + \sum_{k=2}^{\infty} \left[k^2 - k - 4\right] a_k x^k$

4l. $2a_2 + 6a_3 x + \sum_{n=2}^{\infty} \left[(n+2)(n+1)a_{n+2} - 3a_{n-2}\right] x^n$

5a. $\displaystyle\sum_{k=0}^{\infty}(k+1)x^k \quad \text{for} \quad |x| < 1$

5b. $\displaystyle\sum_{k=0}^{\infty}\frac{1}{2}(k+2)(k+1)x^k \quad \text{for} \quad |x| < 1$

6a. $\displaystyle\sum_{k=0}^{\infty}\frac{1}{k!}x^k$

6b. $\displaystyle\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}x^{2k}$

6c. $\displaystyle\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}x^{2k+1}$

6d. $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}(x-1)^k$

7a. $1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{16}x^3-\frac{5}{128}x^4$

7b i. $1-\frac{1}{2}x+\frac{3}{8}x^2-\frac{5}{16}x^3$

7b ii. $1-\frac{1}{2}x^2+\frac{3}{8}x^4-\frac{5}{16}x^6$

8b i. $1+2x+2x^2+\frac{4}{3}x^4+\frac{4}{15}x^5+\frac{4}{45}x^6+\frac{8}{315}x^7+\frac{2}{315}x^8+\frac{4}{2835}x^9$

8b ii. $1+\frac{1}{2}x^2+\frac{5}{24}x^4+\frac{61}{720}x^6+\frac{277}{8064}x^8+\frac{50521}{3628800}x^{10}$

8b iii. $3-\frac{4}{3}(x-2)+\frac{1}{27}(x-2)^2-\frac{4}{243}(x-2)^3+\frac{31}{4374}(x-2)^4-\frac{58}{19683}(x-2)^5+\frac{139}{118098}(x-2)^6-\frac{238}{531441}(x-2)^7$

9a. $\displaystyle\sum_{k=0}^{\infty}2^k x^k$ with $R = 1/2$

9b. $\displaystyle\sum_{k=0}^{\infty}(-1)^k x^{2k}$ with $R = 1$

9c. $\displaystyle\sum_{k=0}^{\infty}\frac{1}{2^k}x^k$ with $R = 2$

9d. $\displaystyle\sum_{k=0}^{\infty}\frac{k+1}{2^{k+1}}x^k$ with $R = 2$

9e. $\displaystyle\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}x^{2k}$ with $R = \infty$

9f. $\displaystyle\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}x^{4k+2}$ with $R = \infty$

10d. Taylor series $= 0$

10e. Because $f(x)$ does not equal its Taylor series about 0 except right at $x = 0$.