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Power Series Solutions II: Generalizations and Theory

A major goal in this chapter is to confirm the claims made in theorems 31.2 and 31.6 regarding the validity of the algebraic method. Along the way, we will also expand both the set of differential equations for which this method can be considered and our definitions of "regular" and "singular" points. As a bonus, we'll also obtain formulas that, at least in some cases, can simplify the computation of the terms of the power series solutions.

32.1 Equations with Analytic Coefficients

In the previous chapter, we discussed an algebraic method for finding a general power series solution about a point x_0 to any differential equation of the form

$$A(x)y' + B(x)y = 0 \qquad \text{or} \qquad A(x)y'' + B(x)y' + C(x)y = 0$$

where $A(x)$, $B(x)$ and $C(x)$ are polynomials with $A(x_0) \neq 0$. Observe that these polynomials can be written as

$$A(x) = \sum_{k=0}^{N} a_k (x - x_0)^k \qquad \text{with} \quad a_0 \neq 0 \quad ,$$

$$B(x) = \sum_{k=0}^{N} b_k (x - x_0)^k \qquad \text{and} \qquad C(x) = \sum_{k=0}^{N} c_k (x - x_0)^k$$

where N is the highest power appearing in these polynomials. Now, I know just what you are wondering: *Must N be finite? Or will our algebraic method still work if $N = \infty$?* That is, can we use our algebraic method to find power series solutions about x_0 to

$$A(x)y' + B(x)y = 0 \qquad \text{and} \qquad A(x)y'' + B(x)y' + C(x)y = 0$$

when $A(x)$, $B(x)$ and $C(x)$ are functions expressible as power series about x_0 (i.e., when A, B and C are functions analytic at x_0), with $A(x_0) \neq 0$.

And the answer to this question is *yes*, at least in theory. Simply replace the coefficients in the differential equations with their power series about x_0, and follow the steps already outlined in sections 31.2 and 31.4 (possibly using the formula from theorem 30.12 on page 30–20 for multiplying infinite series).

There are, of course, some further questions you are bound to be asking regarding these power series solutions and the finding of them. In particular:

1. What will be the radii of convergence for the resulting power series solutions?

and

2. Are there any shortcuts to what could clearly be a rather lengthy and tedious set of calculations.

For the answers, read on.

32.2 Ordinary and Singular Points, the Radius of Analyticity, and the Reduced Form

Introducing Complex Variables

To properly address at least one of our questions, and to simplify the statements of our theorems, it will help to start viewing the coefficients of our differential equations as functions of a complex variable z. We actually did this in the last chapter when we referred to a point z_s in the complex plane for which $A(z_s) = 0$. But A was a polynomial then, and viewing polynomials as functions of a complex variable is so easy that we hardly noted doing so. Viewing other functions (such as exponentials, logarithms and trigonometric functions) as functions of a complex variable may be a bit more challenging.

Analyticity and Power Series

Let us start by recalling that we need not restrict the variable or the center in a power series to real values — they can be complex,

$$\sum_{k=0}^{\infty} a_k (z - z_0)^k \qquad \text{for} \quad |z - z_0| < R \quad ,$$

in which case the radius of convergence R is the radius of the largest open disk in the complex plane centered at z_0 on which the power series is convergent.[1]

Also recall that our definition of analyticity also applies to functions of a complex variable; that is, any function f of a complex variable is *analytic* at a point z_0 in the complex plane if and only if $f(z)$ can be expressed as a power series about z_0,

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k \qquad \text{for} \quad |z - z_0| < R$$

for some $R > 0$. Moreover, as also noted in section 30.3, if f is any function of a real variable given by a power series on the interval $(x_0 - R, x_0 + R)$,

$$f(x) = \sum_{k=0}^{\infty} a_k (x - x_0)^k \quad ,$$

[1] If you don't recall this, quickly review section 30.3.

then we can view this function as a function of the complex variable $z = x + iy$ on a disk of radius R about x_0 by simply replacing the real variable x with the complex variable z,

$$f(z) = \sum_{k=0}^{\infty} a_k (z - x_0)^k \quad .$$

We will do this automatically in all that follows.

By the way, do observe that, if

$$\lim_{z \to z_0} |f(z)| = \infty \quad ,$$

then f certainly is not analytic at z_0 !

Some Results from Complex Analysis

Useful insights regarding analytic functions can be gained from the theory normally developed in an introductory course on "complex analysis". Sadly, we do not have the time or space to properly develop that theory here. As an alternative, a brief overview of the relevant parts of that theory is given for the interested reader in an appendix near the end of this chapter (section 32.7). From that appendix, we get the following two lemmas (both of which should seem reasonable):

Lemma 32.1
Assume F is a function analytic at z_0 with corresponding power series $\sum_{k=0}^{\infty} f_k (z - z_0)^k$, and let R be either some positive value or $+\infty$. Then

$$F(z) = \sum_{k=0}^{\infty} f_k (z - z_0)^k \qquad \textit{whenever} \quad |z - z_0| < R$$

if and only if F is analytic at every complex point z satisfying

$$|z - z_0| < R \quad .$$

Lemma 32.2
Assume $F(z)$ and $A(z)$ are two functions analytic at a point z_0 . Then the quotient $^F/_A$ is also analytic at z_0 if and only if

$$\lim_{z \to z_0} \frac{F(z)}{A(z)}$$

is finite.

Let us note the following immediate corollary of the first lemma:

Corollary 32.3
Assume $F(x)$ is some function on the real line, and $\sum_{k=0}^{\infty} f_k (x - x_0)^k$ is a power series with a infinite radius of convergence . If

$$F(x) = \sum_{k=0}^{\infty} f_k (x - x_0)^k \qquad \textit{for} \quad -\infty < x < \infty \quad ,$$

then $F(z)$ is analytic at every point in the complex plane.

From this corollary and the series in set (30.4) on page 30–7, it immediately follows that the sine and cosine functions, as well as the exponential functions are all analytic on the entire complex plane.

Let us also note that the second lemma extends some observations regarding quotients of functions made in section 30.4.[2]

Ordinary and Singular Points

Let z_0 be a point on the complex plane, and let a, b and c be functions on the complex plane. We will say that z_0 is an *ordinary point* for the first-order differential equation

$$a(x)y' + b(x)y = 0$$

if and only if the quotient

$$\frac{b(z)}{a(z)}$$

is analytic at z_0. And we will say that z_0 is an ordinary point for the second-order differential equation

$$a(x)y'' + b(x)y' + c(x)y = 0$$

if and only if the quotients

$$\frac{b(z)}{a(z)} \quad \text{and} \quad \frac{c(z)}{a(z)}$$

are both analytic at z_0.

Any point that is not an ordinary point (that is, any point at which the above quotients are not analytic) is called a *singular point* for the differential equation.

Using lemma 32.2, you can easily verify the following shortcuts for determining whether a point is a singular or ordinary point for a given differential equation. You can then use these lemmas to verify that our new definitions reduce to those given in the last chapter when the coefficients of our differential equation are rational functions.

Lemma 32.4

Let z_0 be a point in the complex plane, and consider the differential equation

$$a(x)y' + b(x)y = 0$$

where a and b are functions analytic at z_0. Then:

1. *If $a(z_0) \neq 0$, then z_0 is an ordinary point for the differential equation.*
2. *If $a(z_0) = 0$ and $b(z_0) \neq 0$, then z_0 is a singular point for the differential equation.*
3. *The point z_0 is an ordinary point for this differential equation if and only if*

$$\lim_{z \to z_0} \frac{b(z)}{a(z)}$$

is finite.

[2] If you haven't already done so, now might be a good time to at least skim over the material in the subsection *More on Algebra with Power Series and Analytic Functions* starting on page 30–20.

Lemma 32.5

Let z_0 be a point in the complex plane, and consider the differential equation

$$a(x)y'' + b(x)y' + c(x)y = 0$$

where a , b and c are functions analytic at z_0 . Then:

1. *If $a(z_0) \neq 0$, then z_0 is an ordinary point for the differential equation.*
2. *If $a(z_0) = 0$, and either $b(z_0) \neq 0$ or $c(z_0) \neq 0$, then z_0 is a singular point for the differential equation.*
3. *The point z_0 is an ordinary point for this differential equation if and only if*

   $$\lim_{z \to z_0} \frac{b(z)}{a(z)} \qquad \text{and} \qquad \lim_{z \to z_0} \frac{c(z)}{a(z)}$$

   are both finite.

!►*Example 32.1:* *Consider the two differential equations*

$$y'' + \sin(x)\,y = 0 \qquad \text{and} \qquad \sin(x)\,y'' + y = 0 \quad .$$

From corollary 32.3, we know that the sine function is analytic at every point on the complex plane, and that

$$\sin(z) = 0 \qquad \text{if} \quad z = n\pi \quad \text{with} \quad n = 0, \pm 1, \pm 2, \ldots \quad .$$

Moreover, it's not hard to show (see exercise 32.5) that the above points are the only points in the complex plane at which the sine is zero.

What this means is that both coefficients of

$$y'' + \sin(x)\,y = 0$$

are analytic everywhere, with the first coefficient (which is simply the constant 1) never being zero. Thus, lemma 32.5 assures us that every point in the complex plane is an ordinary point for this differential equation. It has no singular points.

On the other hand, while both coefficients of

$$\sin(x)\,y'' + 5y = 0$$

are analytic everywhere, the first coefficient is zero at $z_0 = 0$ (and at every other integral multiple of π). Since the second coefficient (again, the constant 1) is not zero at $z_0 = 0$, lemma 32.5 tells us that $z_0 = 0$ (and every other integral multiple of π) is a singular point for this differential equation.

Radius of Analyticity

The Definition, Recycled

Why waste a perfectly good definition? Given

$$a(x)y' + b(x)y = 0 \qquad \text{or} \qquad a(x)y'' + b(x)y' + c(x)y = 0$$

we define the *radius of analyticity* (for the differential equation) about any given point z_0 to be the distance between z_0 and the singular point closest to z_0, unless the differential equation has no singular points, in which case we define the *radius of analyticity* to be $+\infty$.

This is precisely the same definition as given (twice) in the previous chapter.

Is the Radius Well Defined?

When the coefficients of our differential equations were just polynomials, it should have been obvious that there really was a "singular point closest to z_0" (provided the equation had singular points). But a cynical reader — especially one who has seen some advanced analysis — may wonder if such a singular point always exists with our more general equations, or if, instead, a devious mathematician could construct a differential equation with an infinite set of singular points, none of which are closest to the given ordinary point. Don't worry, no mathematician is devious enough.

Lemma 32.6
Let z_0 be an ordinary point for some first- or second-order linear homogeneous differential equation. Then, if the differential equation has singular points, there is at least one singular point z_s such that no other singular point is closer to z_0 than z_s.

The z_s in this lemma is a "singular point closest to z_0". There may, in fact, be other singular points at the same distance from z_0, but none closer. Anyway, this ensures that "the radius of analyticity" for a given differential equation about a given point is well defined.

The proof of lemma 32.6 is subtle, and is discussed in an appendix (section 32.8).

32.3 The Reduced Forms

A Standard Way to Rewrite Our Equations

There is some benefit in dividing a given differential equation

$$ay' + by = 0 \qquad \text{or} \qquad ay'' + by' + cy = 0$$

by the equation's leading coefficient, obtaining the equation's corresponding *reduced form*[3]

$$y' + Py = 0 \qquad \text{or} \qquad y'' + Py' + Qy = 0$$

(with $P = b/a$ and $Q = c/a$). For one thing, it may reduce the number of products of infinite series to be computed. In addition, it will allow us to use the generic recursion formulas that

[3] also called the *normal form* by some authors

we will be deriving in a little bit. However, the advantages of using the reduced form depends somewhat on the ease in finding and using the power series for P (and, in the second-order case, for Q). If the differential equation can be written as

$$Ay' + By = 0 \qquad \text{or} \qquad Ay'' + By' + Cy = 0$$

where the coefficients are given by relatively simple known power series, then the extra effort in finding and using the power series for the coefficients of the corresponding reduced equations

$$y' + Py = 0 \qquad \text{or} \qquad y'' + Py' + Qy = 0$$

may out-weigh any supposed advantages of using these reduced forms. In particular, if A , B and C are all relatively simple polynomials (with A not being a constant), then dividing

$$Ay'' + By' + Cy = 0$$

by A is unlikely to simplify your computations — don't do it unless ordered to do so in an exercise.

Ordinary Points and the Reduced Form

The next two lemmas will be important in deriving the general formulas for power series solutions. However, they follow almost immediately from lemmas 32.1 and 32.2, along with the definitions of "reduced form", "regular and singular points", "radius of convergence" and "analyticity at z_0 ".

Lemma 32.7
Let

$$ay' + by = 0$$

have reduced form

$$y' + Py = 0 \quad .$$

Then z_0 is an ordinary point for this differential equation if and only if P is analytic at z_0 . Moreover, if z_0 is an ordinary point, then P has a power series representation

$$P(z) = \sum_{k=0}^{\infty} p_k (z - z_0)^k \qquad \text{for} \quad |z - z_0| < R$$

where R is the radius of analyticity for this differential equation about z_0 .

Lemma 32.8
Let

$$ay'' + by' + cy = 0$$

have reduced form

$$y'' + Py' + Qy = 0 \quad .$$

Then z_0 is an ordinary point for this differential equation if and only if both P and Q are analytic at z_0 . Moreover, if z_0 is an ordinary point, then P and Q have power series representations

$$P(z) = \sum_{k=0}^{\infty} p_k (z - z_0)^k \qquad \text{for} \quad |z - z_0| < R$$

and

$$Q(z) = \sum_{k=0}^{\infty} q_k (z - z_0)^k \qquad \text{for} \quad |z - z_0| < R \quad .$$

where R *is the radius of analyticity for this differential equation about* z_0 *.*

32.4 Existence of Power Series Solutions

Deriving the Generic Recursion Formulas

First-Order Case

Let us try to find the general power series solution to

$$y' + Py = 0$$

about $x_0 = 0$ when P is any function analytic at $x_0 = 0$. This analyticity means P has a power series representation

$$P(x) = \sum_{k=0}^{\infty} p_k x^k \qquad \text{for} \quad |x| < R$$

for some $R > 0$. We'll assume that this series and a value for R are known.

Our approach is to follow the method described in section 31.2 as far as possible. We assume y is given by a yet unknown power series about $x_0 = 0$,

$$y(x) = \sum_{k=0}^{\infty} a_k x^k \quad ,$$

compute the corresponding series for y', plug that into the differential equation, and "compute" (using the above series for P and the formula for series multiplication from theorem 30.12 on page 30–20):

$$y' + Py = 0$$

$$\hookrightarrow \qquad \sum_{k=1}^{\infty} k a_k x^{k-1} + \left(\sum_{k=0}^{\infty} a_k x^k\right)\left(\sum_{k=0}^{\infty} p_k x^k\right) = 0$$

$$\hookrightarrow \qquad \underbrace{\sum_{k=1}^{\infty} k a_k x^{k-1}}_{n = k-1} + \underbrace{\sum_{k=0}^{\infty}\left[\sum_{j=0}^{k} a_j p_{k-j}\right] x^k}_{n = k} = 0$$

$$\hookrightarrow \qquad \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n + \sum_{n=0}^{\infty}\left[\sum_{j=0}^{n} a_j p_{n-j}\right] x^n = 0$$

$$\hookrightarrow \qquad \sum_{n=0}^{\infty}\left[(n+1)a_{n+1} + \sum_{j=0}^{n} a_j p_{n-j}\right] x^n = 0 \quad .$$

Thus,

$$(n+1)a_{n+1} \ + \ \sum_{j=0}^{n} a_j p_{n-j} \ = \ 0 \qquad \text{for} \quad n = 0, 1, 2, \ldots \quad .$$

Solving for a_{n+1} and letting $k = n+1$ gives us

$$a_k \ = \ -\frac{1}{k}\sum_{j=0}^{k-1} a_j p_{k-1-j} \qquad \text{for} \quad k = 1, 2, 3, \ldots \quad . \tag{32.1}$$

Of course, we would have obtained the same recursion formula with x_0 being any ordinary point for the given differential equation (just replace x in the above computations with $X = x - x_0$).

Second-Order Case

We will leave this derivation as an exercise.

?►*Exercise 32.1:* *Assume that, over some interval containing the point x_0, P and Q are functions given by power series*

$$P(x) \ = \ \sum_{k=0}^{\infty} p_k (x-x_0)^k \qquad \text{and} \qquad Q(x) \ = \ \sum_{k=0}^{\infty} q_k (x-x_0)^k \quad ,$$

and derive the recursion formula

$$a_k \ = \ -\frac{1}{k(k-1)} \sum_{j=0}^{k-2} \big[(j+1)a_{j+1}p_{k-2-j} \ + \ a_j q_{k-2-j}\big] \tag{32.2}$$

for the series solution

$$y(x) \ = \ \sum_{k=0}^{\infty} a_k (x-x_0)^k$$

to

$$y'' \ + \ Py' \ + \ Qy \ = \ 0 \quad .$$

(For simplicity, start with the case in which $x_0 = 0$.)

Validity of the Power Series Solutions

Here are the big theorems on the existence of power series solutions. They are also theorems on the computation of these solutions since they contain the recursion formulas just derived.

Theorem 32.9 (first-order series solutions)
Suppose x_0 is an ordinary point for a first-order homogeneous differential equation whose reduced form is

$$y' \ + \ Py \ = \ 0 \quad .$$

Then P has a power series representation

$$P(x) \ = \ \sum_{k=0}^{\infty} p_k (x-x_0)^k \qquad \text{for} \quad |x - x_0| < R$$

where R is the radius of analyticity about x_0 for this differential equation.

Moreover, a general solution to the differential equation is given by

$$y(x) = \sum_{k=0}^{\infty} a_k (x - x_0)^k \qquad \text{for} \quad |x - x_0| < R$$

where a_0 is arbitrary, and the other a_k's satisfy the recursion formula

$$a_k = -\frac{1}{k} \sum_{j=0}^{k-1} a_j p_{k-1-j} \quad . \tag{32.3}$$

Theorem 32.10 (second-order series solutions)

Suppose x_0 is an ordinary point for a second-order homogeneous differential equation whose reduced form is

$$y'' + Py' + Qy = 0 \quad .$$

Then P and Q have power series representations

$$P(x) = \sum_{k=0}^{\infty} p_k (x - x_0)^k \qquad \text{for} \quad |x - x_0| < R$$

and

$$Q(x) = \sum_{k=0}^{\infty} q_k (x - x_0)^k \qquad \text{for} \quad |x - x_0| < R$$

where R is the radius of analyticity about x_0 for this differential equation.

Moreover, a general solution to the differential equation is given by

$$y(x) = \sum_{k=0}^{\infty} a_k (x - x_0)^k \qquad \text{for} \quad |x - x_0| < R$$

where a_0 and a_1 are arbitrary, and the other a_k's satisfy the recursion formula

$$a_k = -\frac{1}{k(k-1)} \sum_{j=0}^{k-2} \left[(j+1) a_{j+1} p_{k-2-j} + a_j q_{k-2-j} \right] \quad . \tag{32.4}$$

There are four major parts to the proof of each of these theorems:

1. Deriving the recursion formula. (Done!)

2. Assuring ourselves that the coefficient functions in the reduced forms have the stated power series representations. (Done! See lemmas 32.7 and 32.7.)

3. Verifying that the radius of convergence for the power series generated from the given recursion formula is at least R.

4. Noting that the calculations used to obtain each recursion formula also confirm that the resulting series is the solution to the given differential equation over the interval $(x_0 - R, x_0 + R)$. (So noted!)

Thus, all that remains to proving these two major theorems to verify the claimed radii of convergence for the given series solutions. This verification is not difficult, but is a bit lengthy and technical, and may not be as exciting to the reader as was the derivation of the recursion formulas. Those who are interested should proceed to section 32.5.

But now, let us try using our new theorems.

!► *Example 32.2:* *Consider, again, the differential equation from example 31.7 on page 31–33,*

$$y'' + \cos(x)y = 0 \quad .$$

Again, let us try to find at least a partial sum of the general power series solution about $x_0 = 0$ *. This time, however, we will use the results from theorem 32.10.*

The equation is already in reduced form

$$y'' + Py' + Qy = 0$$

with $P(x) = 0$ *and* $Q(x) = \cos(x)$ *. Since both of these functions are analytic on the entire complex plane, the theorem assures us that there is a general power series solution*

$$y(x) = \sum_{k=0}^{\infty} a_k x^k \qquad \text{for} \quad |x| < \infty$$

with a_0 *and* a_1 *being arbitrary, and with the other* a_k*'s being given through recursion formula (32.4). And to use this recursion formula, we need the corresponding power series representations for* P *and* Q *. The series for* P *, of course, is trivial,*

$$P(x) = 0 \iff P(x) = \sum_{k=0}^{\infty} p_k x^k \qquad \text{with} \quad p_k = 0 \qquad \text{for all} \quad k \quad .$$

Fortunately, the power series for Q *is well-known, and only needs to be slightly rewritten for use in our recursion formula:*

$$\begin{aligned} Q(x) &= \cos(x) \\ &= \sum_{m=0}^{\infty} (-1)^m \frac{1}{(2m)!} x^{2m} \\ &= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots \\ &= (-1)^{0/2}x^0 + 0x^1 + (-1)^{2/2}\frac{1}{2!}x^2 + 0x^3 \\ &\qquad + (-1)^{4/2}\frac{1}{4!}x^4 + 0x^5 + (-1)^{6/2}\frac{1}{6!}x^6 + 0x^7 + \cdots \quad . \end{aligned}$$

That is,

$$Q(x) = \sum_{k=0}^{\infty} q_k x^k \qquad \text{with} \quad q_k = \begin{cases} (-1)^{k/2}\dfrac{1}{k!} & \text{if } k \text{ is even} \\ 0 & \text{if } k \text{ is odd} \end{cases} \quad .$$

Using the above with recursion formula (32.4) gives us, for $k \geq 2$ *,*

$$a_k = -\frac{1}{k(k-1)} \sum_{j=0}^{k-2} \big[(j+1)a_{j+1} \underbrace{p_{k-2-j}}_{0} + a_j q_{k-2-j} \big]$$

$$= -\frac{1}{k(k-1)} \sum_{j=0}^{k-2} a_j \left\{ \begin{array}{cl} (-1)^{(k-2-j)/2} \dfrac{1}{(k-2-j)!} & \text{if } k-2-j \text{ is even} \\ 0 & \text{if } k-2-j \text{ is odd} \end{array} \right\} \quad ,$$

which simplifies, slightly, to

$$a_k = \frac{1}{k(k-1)} \sum_{j=0}^{k-2} a_j \left\{ \begin{array}{cl} (-1)^{(k-j)/2} \dfrac{1}{(k-2-j)!} & \text{if } k-j \text{ is even} \\ 0 & \text{if } k-j \text{ is odd} \end{array} \right\} \quad .$$

Repeatedly using this formula, we see that

$$a_2 = \frac{1}{2(2-1)} \sum_{j=0}^{2-2} a_j \left\{ \begin{array}{cl} (-1)^{(2-j)/2} \dfrac{1}{(2-2-j)!} & \text{if } 2-j \text{ is even} \\ 0 & \text{if } 2-j \text{ is odd} \end{array} \right\}$$

$$= \frac{1}{2} a_0 \left\{ \begin{array}{cl} (-1)^{(2-0)/2} \dfrac{1}{(2-2-0)!} & \text{if } 2-0 \text{ is even} \\ 0 & \text{if } 2-0 \text{ is odd} \end{array} \right\}$$

$$= \frac{1}{2} a_0 \{-1\}$$

$$= -\frac{1}{2} a_0 \quad ,$$

$$a_3 = \frac{1}{3(3-1)} \sum_{j=0}^{3-2} a_j \left\{ \begin{array}{cl} (-1)^{(3-j)/2} \dfrac{1}{(3-2-j)!} & \text{if } 3-j \text{ is even} \\ 0 & \text{if } 3-j \text{ is odd} \end{array} \right\}$$

$$= \frac{1}{3 \cdot 2} \left[a_0 \left\{ \begin{array}{cl} (-1)^{(3-0)/2} \dfrac{1}{(3-2-0)!} & \text{if } 3-0 \text{ is even} \\ 0 & \text{if } 3-0 \text{ is odd} \end{array} \right\} \right.$$

$$\left. + \; a_1 \left\{ \begin{array}{cl} (-1)^{(3-1)/2} \dfrac{1}{(3-2-1)!} & \text{if } 3-1 \text{ is even} \\ 0 & \text{if } 3-1 \text{ is odd} \end{array} \right\} \right]$$

$$= \frac{1}{3 \cdot 2} [a_0 \cdot 0 + a_1(-1)]$$

$$= -\frac{1}{3 \cdot 2} a_1 \quad ,$$

$$a_4 = \frac{1}{4(4-1)} \sum_{j=0}^{4-2} a_j \left\{ \begin{array}{cl} (-1)^{(4-j)/2} \dfrac{1}{(4-2-j)!} & \text{if } 4-j \text{ is even} \\ 0 & \text{if } 4-j \text{ is odd} \end{array} \right\}$$

$$= \frac{1}{4 \cdot 3} \left[a_0 \begin{Bmatrix} (-1)^{(4-0)/2} \frac{1}{(4-2-0)!} & \text{if } 4-0 \text{ is even} \\ 0 & \text{if } 4-0 \text{ is odd} \end{Bmatrix} \right.$$

$$+ \; a_1 \begin{Bmatrix} (-1)^{(4-1)/2} \frac{1}{(4-2-1)!} & \text{if } 4-1 \text{ is even} \\ 0 & \text{if } 4-1 \text{ is odd} \end{Bmatrix}$$

$$\left. + \; a_2 \begin{Bmatrix} (-1)^{(4-2)/2} \frac{1}{(4-2-2)!} & \text{if } 4-2 \text{ is even} \\ 0 & \text{if } 4-2 \text{ is odd} \end{Bmatrix} \right]$$

$$= \frac{1}{4 \cdot 3}\left[a_0 \frac{1}{2!} \;+\; 0 \;-\; a_2\right]$$

$$= \frac{1}{4 \cdot 3}\left[\frac{1}{2}a_0 \;+\; 0 \;-\; \left(-\frac{1}{2}a_0\right)\right]$$

$$= \frac{2}{4!}a_0 \quad ,$$

$$a_5 \;=\; \frac{1}{5(5-1)} \sum_{j=0}^{5-2} a_j \begin{Bmatrix} (-1)^{(5-j)/2} \frac{1}{(5-2-j)!} & \text{if } 5-j \text{ is even} \\ 0 & \text{if } 5-j \text{ is odd} \end{Bmatrix}$$

$$= \frac{1}{5 \cdot 4} \left[a_0 \begin{Bmatrix} (-1)^{(5-0)/2} \frac{1}{(5-2-0)!} & \text{if } 5-0 \text{ is even} \\ 0 & \text{if } 5-0 \text{ is odd} \end{Bmatrix} \right.$$

$$+ \; \cdots$$

$$\left. + \; a_3 \begin{Bmatrix} (-1)^{(5-3)/2} \frac{1}{(5-2-3)!} & \text{if } 5-3 \text{ is even} \\ 0 & \text{if } 5-3 \text{ is odd} \end{Bmatrix} \right]$$

$$= \frac{1}{5 \cdot 4}\left[a_0 \cdot 0 \;+\; a_1 \cdot \frac{1}{2} \;+\; a_2 \cdot 0 \;+\; a_3(-1)\right]$$

$$= \frac{1}{5 \cdot 4}\left[0 \;+\; \frac{1}{2}a_1 \;+\; 0 \;-\; \left(-\frac{1}{3 \cdot 2}a_1\right)\right]$$

$$= \frac{4}{5!}a_1 \quad ,$$

and

$$a_6 \;=\; \frac{1}{6(6-1)} \sum_{j=0}^{6-2} a_j \begin{Bmatrix} (-1)^{(6-j)/2} \frac{1}{(6-2-j)!} & \text{if } 6-j \text{ is even} \\ 0 & \text{if } 6-j \text{ is odd} \end{Bmatrix}$$

$$= \cdots$$

$$= \frac{1}{6 \cdot 5}\left[-\frac{1}{4!}a_0 \;+\; 0a_1 \;+\; \frac{1}{2!}a_2 \;+\; 0a_3 - a_4\right]$$

$$= \frac{1}{6 \cdot 5}\left[-\frac{1}{4!}a_0 \;+\; 0 \;+\; \frac{1}{2!}\left(-\frac{1}{2}a_0\right) \;+\; 0 - \left(\frac{2}{4!}a_0\right)\right]$$

$$= -\frac{9}{6!}a_0 \quad .$$

Thus, the sixth partial sum of the power series for y *about* 0 *is*

$$\begin{aligned} S_6(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \\ &= a_0 + a_1 x + \frac{-a_0}{2} x^2 + \frac{-a_1}{3!} x^3 + \frac{2a_0}{4!} x^4 + \frac{4a_1}{5!} x^5 + \frac{-9a_0}{6!} x^6 \\ &= a_0 \left[1 - \frac{1}{2} x^2 + \frac{1}{12} x^4 - \frac{1}{80} x^6 \right] + a_1 \left[x - \frac{1}{6} x^3 + \frac{1}{30} x^5 \right] \quad , \end{aligned}$$

just as we had found, using the Taylor series method, in example 31.7 on page 31–33.

If you compare the work done in the last example with the work done in example 31.7, it may appear that, while we obtained identical results, we expended much more work in using the recursion formula from theorem 32.10 than in using the Taylor series method. On the other hand, all the computations done in the last example were fairly simple arithmetic computations — computations that we could have easily programmed a computer to do. So there can be computational advantages to using our new results.

You also certainly noticed that a few computations were skipped over. You do them.

?►Exercise 32.2: *Fill in the missing details in the computations of* a_5 *and* a_6 *in the last example.*

32.5 The Radius of Convergence for the Solution Series

To finish our proofs of theorems 32.10 and 32.9, we need to verify that the radius of convergence for each of the given series solutions is at least the given value for R. We will do this for the solution series in theorem 32.10, and leave the corresponding verification for theorem 32.9 (which will be slightly easier) as an exercise.

What We Have, and What We Need to Show

Recall: We have a positive value R and two power series

$$\sum_{k=0}^{\infty} p_k X^k \qquad \text{and} \qquad \sum_{k=0}^{\infty} q_k X^k$$

that we know converge when $|X| < R$ (for simplicity, we're letting $X = x - x_0$). We also have a corresponding power series

$$\sum_{k=0}^{\infty} a_k X^k$$

where a_0 and a_1 are arbitrary, and the other coefficients are given by the recursion formula

$$a_k = -\frac{1}{k(k-1)} \sum_{j=0}^{k-2} \left[(j+1) a_{j+1} p_{k-2-j} + a_j q_{k-2-j} \right] \qquad \text{for} \quad k = 2, 3, 4, \ldots \quad .$$

We now only need to show that $\sum_{k=0}^{\infty} a_k X^k$ converges whenever $|X| < R$, and to do that, we will produce another power series $\sum_{k=0}^{\infty} b_k X^k$ whose convergence is "easily" shown using the limit ratio test, and which is related to our first series by

$$|a_k| \leq b_k \qquad \text{for} \quad k = 0, 1, 2, 3, \ldots \quad .$$

By the comparison test, it then immediately follows that $\sum_{k=0}^{\infty} \left| a_k X^k \right|$, and hence also $\sum_{k=0}^{\infty} a_k X^k$, converges.

So let X be any value with $|X| < R$.

Constructing the Series for Comparison

Our first step in constructing $\sum_{k=0}^{\infty} b_k X^k$ is to pick some value r between $|X|$ and R,

$$0 \leq |X| < r < R \quad .$$

Since $|r| < R$, we know the series

$$\sum_{k=0}^{\infty} p_k r^k \qquad \text{and} \qquad \sum_{k=0}^{\infty} q_k r^k$$

both converge. But a series cannot converge if the terms in the series become arbitrarily large in magnitude. So the magnitudes of these terms — the $\left| p_k r^k \right|$'s and $\left| q_k r^k \right|$'s — must be bounded; that is, there must be a finite number M such that

$$\left| p_k r^k \right| < M \qquad \text{and} \qquad \left| q_k r^k \right| < M \qquad \text{for} \quad k = 0, 1, 2, 3, \ldots \quad .$$

Equivalently (since $r > 0$),

$$|p_k| < \frac{M}{r^k} \qquad \text{and} \qquad |q_k| < \frac{M}{r^k} \qquad \text{for} \quad k = 0, 1, 2, 3, \ldots \quad .$$

These inequalities, the triangle inequality and the recursion formula combine to give us, for $k = 2, 3, 4, \ldots$,

$$\begin{aligned}
|a_k| &= \left| -\frac{1}{k(k-1)} \sum_{j=0}^{k-2} \left[(j+1) a_{j+1} p_{k-2-j} \; + \; a_j q_{k-2-j} \right] \right| \\
&\leq \frac{1}{k(k-1)} \sum_{j=0}^{k-2} \left[(j+1) \left| a_{j+1} \right| \left| p_{k-2-j} \right| \; + \; \left| a_j \right| \left| q_{k-2-j} \right| \right] \\
&\leq \frac{1}{k(k-1)} \sum_{j=0}^{k-2} \left[(j+1) \left| a_{j+1} \right| \frac{M}{r^{k-2-j}} \; + \; \left| a_j \right| \frac{M}{r^{k-2-j}} \right] \quad ,
\end{aligned}$$

which we will rewrite as

$$|a_k| \;\leq\; \frac{1}{k(k-1)} \sum_{j=0}^{k-2} \frac{M\left[(j+1)\left|a_{j+1}\right| + \left|a_j\right|\right]}{r^{k-2-j}} \quad .$$

Now let $b_0 = |a_0|$, $b_1 = |a_1|$ and

$$b_k \;=\; \sum_{j=0}^{k-2} \frac{M\left[(j+1)\left|a_{j+1}\right| + \left|a_j\right|\right]}{r^{k-2-j}} \qquad \text{for} \quad k = 2, 3, 4, \ldots \quad .$$

From the preceding inequality, it is clear that we've chosen the b_k's so that

$$|a_k| \;\leq\; b_k \qquad \text{for} \quad k = 0, 1, 2, 3, \ldots \quad .$$

In fact, we even have

$$|a_k| \;\leq\; \frac{1}{k(k-1)} b_k \qquad \text{for} \quad k = 2, 3, \ldots \quad . \tag{32.5}$$

Thus,

$$\left|a_k X^k\right| \;\leq\; b_k \left|X\right|^k \qquad \text{for} \quad k = 2, 3, \ldots \quad ,$$

and (by the comparison text) we can confirm the convergence of $\sum_{k=0}^{\infty} a_k X^k$ by simply verifying the convergence of $\sum_{k=0}^{\infty} b_k |X|^k$.

Convergence of the Comparison Series

According to the limit convergence test (theorem 30.2 on page 30–6), $\sum_{k=0}^{\infty} b_k |X|^k$ converges if

$$\lim_{k\to\infty} \left| \frac{b_{k+1} X^{k+1}}{b_k X^k} \right| \;<\; 1 \quad .$$

Well, let $k > 2$. Using the formula for the b_k's with k replaced with $k+1$, we get

$$\begin{aligned}
b_{k+1} \;&=\; \sum_{j=0}^{[k+1]-2} \frac{M\left[(j+1)\left|a_{j+1}\right| + \left|a_j\right|\right]}{r^{[k+1]-2-j}} \\
&=\; \sum_{j=0}^{k-1} \frac{M\left[(j+1)\left|a_{j+1}\right| + \left|a_j\right|\right]}{r^{k-1-j}} \\
&=\; \sum_{j=0}^{k-2} \frac{M\left[(j+1)\left|a_{j+1}\right| + \left|a_j\right|\right]}{r^{k-1-j}} \;+\; \frac{M\left[([k-1]+1)\left|a_{[k-1]+1}\right| + \left|a_{[k-1]}\right|\right]}{r^{k-1-[k-1]}} \\
&=\; \frac{1}{r} \sum_{j=0}^{k-2} \frac{M\left[(j+1)\left|a_{j+1}\right| + \left|a_j\right|\right]}{r^{k-2-j}} \;+\; M\left[k\left|a_k\right| + \left|a_{k-1}\right|\right] \\
&=\; \frac{1}{r} b_k \;+\; kM\left|a_k\right| \;+\; M\left|a_{k-1}\right|
\end{aligned}$$

But, by inequality (32.5),

$$kM\left|a_k\right| \;\leq\; kM \frac{1}{k(k-1)} b_k \;=\; \frac{M}{k-1} b_k \quad . \tag{32.6}$$

Moreover, because the terms in the summation for b_k are all nonnegative real numbers,

$$\begin{aligned} b_k &= \sum_{j=0}^{k-2} \frac{M\left[(j+1)\left|a_{j+1}\right| + \left|a_j\right|\right]}{r^{k-2-j}} \\ &\geq \text{ the last term in the summation} \\ &= \frac{M(j+1)\left|a_{j+1}\right|}{r^{k-2-j}} \quad \text{with} \quad j = k-2 \\ &= \frac{M([k-2]+1)\left|a_{[k-2]+1}\right|}{r^{k-2-[k-2]}} \\ &= M(k-1)\left|a_{k-1}\right| \quad . \end{aligned}$$

Thus,

$$M\left|a_{k-1}\right| \leq \frac{1}{(k-1)} b_k \quad . \tag{32.7}$$

Combining inequalities (32.6) and (32.7) with the last formula above for b_{k+1} gives us

$$\begin{aligned} b_{k+1} &= \frac{1}{r} b_k + kM\left|a_k\right| + M\left|a_{k-1}\right| \\ &\leq \frac{1}{r} b_k + \frac{M}{(k-1)} b_k + \frac{1}{(k-1)} b_k = \left[\frac{1}{r} + \frac{M+1}{k-1}\right] b_k \quad . \end{aligned}$$

That is,

$$\frac{b_{k+1}}{b_k} \leq \frac{1}{r} + \frac{M+1}{k-1} \quad .$$

From this and the fact that $|X| < r$, we see that

$$\lim_{k\to\infty} \left|\frac{b_{k+1}X^{k+1}}{b_k X^k}\right| = \lim_{k\to\infty} \left[\frac{1}{r} + \frac{M+1}{k-1}\right] |X| = \left[\frac{1}{r} + 0\right] |X| = \frac{|X|}{r} < 1 \quad ,$$

confirming (by the limit ratio test) that $\sum_{k=0}^{\infty} b_k X^k$ converges, and, thus, completing our proof of theorem 32.10. ∎

To finish the proof of theorem 32.9, do the following exercise:

?►*Exercise 32.3:* *Let* $\sum_{k=0}^{\infty} p_k X^k$ *be a power series that converges for* $|X| < R$*, and let* $\sum_{k=0}^{\infty} a_k X^k$ *be a power series where* a_0 *is arbitrary, and the other coefficients are given by the recursion formula*

$$a_k = -\frac{1}{k} \sum_{j=0}^{k-1} a_j p_{k-1-j} \qquad \text{for} \quad k = 1, 2, 3, \ldots \quad .$$

Show that $\sum_{k=0}^{\infty} a_k X^k$ *converges also for* $|X| < R$*.*

(Suggestion: Go back to the start of this section and "redo" the computations step by step, making the obvious modifications to deal with the given recursion formula.)

32.6 Singular Points and the Radius of Convergence

In the last section, we verified that the power series solutions obtained in this chapter are valid at least over $(x_0 - R, x_0 + R)$ where R is the distance from x_0 to the nearest singular point, provided the differential equation has singular points. This fact can be refined by the following theorem:

Theorem 32.11
Let

$$y(x) = \sum_{k=0}^{\infty} c_k (x - x_0)^k \qquad \text{for} \quad |x - x_0| < R$$

be a power series solution for some first- or second-order homogeneous linear differential equation. Assume, further, that R is finite and is the radius of convergence for the above power series. Then this differential equation has a singular point z_s with $|z_s - x_0| = R$.

Note that we are not requiring x_0 to be an ordinary point for the differential equation. As we've seen in examples, it is possible to have a solution analytic at a singular point for a differential equation.

The proof of this theorem, unfortunately, is nontrivial. The adventurous can read about it in an appendix, section 32.9 (after reading sections 32.7 and 32.8). By the way, the above theorem is actually a consequence of more general results obtained in the appendix, some of which will be useful in some of our more advanced work in chapter 34.

32.7 Appendix: A Brief Overview of Complex Calculus

To properly address issues regarding the analyticity of our functions and the regions of convergence of their power series, we need to delve deeper into the theory of analytic functions — much deeper than normally presented in elementary calculus courses. Instead, we want the theory normally developed in introductory courses in complex analysis. That's because the complex-variable theory exposes a much closer relation between "differentiability" and "analyticity" than does the real-variable theory developed in elementary calculus. If you've had such a course, good; the following will be a review. If you've not had such a course, think about taking one, and read on. What follows is a brief synopsis of the relevant concepts and results from such a course, written assuming you have not had such a course (but have, at least, skimmed the introductory section on complex variables in section 30.3, starting on page 30–16).

Functions of a Complex Variable

In "complex analysis", the basic concepts and theories developed in elementary calculus are extended so that they apply to complex-valued functions of a complex variable. Thus, for example, where we may have considered the "real" polynomial and "real" exponential

$$p(x) = 3x^2 + 4x - 5 \qquad \text{and} \qquad h(x) = e^x \qquad \text{for all } x \text{ in } \mathbb{R}$$

in elementary calculus, in complex analysis we consider the "complex" polynomial and "complex" exponential

$$p(z) = 3z^2 + 4z - 5 \quad \text{and} \quad h(z) = e^z \quad \text{for all } z = x + iy \text{ in } \mathbb{C} \quad .$$

Note that we treat z as a single entity. Still, the complex variable z is just $x+iy$. Consequently, much of complex analysis follows from what you already know about the calculus of functions of two variables. In particular, the partial derivatives with respect to x and y are defined just as they were defined back in your calculus course (and section 3.7 of this text), and when we say f is *continuous at* z_0, we mean that

$$f(z_0) = \lim_{z \to z_0} f(z)$$

with the understanding that

$$\lim_{z \to z_0} f(z) = \lim_{\substack{x \to x_0 \\ y \to y_0}} f(x+iy) \quad \text{with} \quad z_0 = x_0 + iy_0 \quad .$$

Along these lines, you should be aware that, in complex variables, we normally assume that functions are defined over subregions of the complex plane, instead of subintervals of the real line. In what follows, we will often require our region of interest to be *open* (as discussed in section 3.7). For example, we will often refer to the disk of all z satisfying $|z - z_0| < R$ for some complex point z_0 and positive value R. Any such disk is an open region.

Complex Differentiability

Given a function f and a point $z_0 = x_0 + iy_0$ in the complex plane, the *complex derivative of* f *at* z_0 — denoted by $f'(z_0)$ or $\left.{}^{df}\!/_{dz}\right|_{z_0}$ — is given by

$$f'(z_0) = \left.\frac{df}{dz}\right|_{z_0} = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \quad .$$

If this limit exists as a finite complex number, we will say that f is *differentiable with respect to the complex variable at* z_0 (*complex differentiable* for short). Remember, $z = x + iy$; so, for the above limit to make sense, the formula for f must be such that $f(x+iy)$ makes sense for every $x + iy$ in some open region about z_0.

We further say that f is complex differentiable on a region of the complex plane if and only if it is complex differentiable at every point in the region.

Naturally, we can extend the complex derivative to higher orders:

$$f'' = \frac{d^2 f}{dz^2} = \frac{d}{dz}\frac{df}{dz} \quad , \quad f''' = \frac{d^3 f}{dz^3} = \frac{d}{dz}\frac{d^2 f}{dz^2} \quad , \quad \cdots \quad .$$

As with functions of a real variable, if $f^{(k)}$ exists for every positive integer k (at a point or in a region), then we say f is *infinitely differentiable* (at the point or in the region).

In many ways, the complex derivative is analogous to derivative you learned in elementary calculus (the *real-variable derivative*). The same basic basic computational formulas apply, giving us, for example,

$$\frac{d}{dz}z^k = kz^{k-1} \quad , \quad \frac{d}{dz}e^{\alpha z} = \alpha e^{\alpha z} \quad \text{and} \quad \frac{d}{dz}[\alpha f(z) + \beta g(z)] = \alpha f'(z) + \beta g'(z) \quad .$$

In addition, the well-known product and quotient rules can easily be verified, and, in verifying these rules, you automatically verify the following:

Theorem 32.12

Assume f and g are complex differentiable on some open region of the complex plane. Then their product fg is also complex differentiable on that region. Moreover, so is their quotient f/g, provided $g(z) \neq 0$ for every z in this region.

Testing for Complex Diffentiability

If f is complex differentiable in some open region of the complex plane, then, unsurprisingly, the chain rule can be shown to hold. In particular,

$$\frac{\partial}{\partial x} f(x+iy) = f'(x+iy) \cdot \frac{\partial}{\partial x}[x+iy] = f'(x+iy) \cdot 1 = f'(x+iy)$$

and

$$\frac{\partial}{\partial y} f(x+iy) = f'(x+iy) \cdot \frac{\partial}{\partial y}[x+iy] = f'(x+iy) \cdot i = if'(x+iy) \quad .$$

Combining these two equations, we get

$$\frac{\partial}{\partial y} f(x+iy) = if'(x+iy) = i\frac{\partial}{\partial x} f(x+iy) \quad .$$

Thus, if f is complex differentiable in some open region, then

$$\frac{\partial}{\partial y} f(x+iy) = i\frac{\partial}{\partial x} f(x+iy) \tag{32.8}$$

at every point $z = x+iy$ in that region.[4] Right off, this gives us a test for "nondifferentiability": If equation (32.8) does not hold throughout some region, then f is not complex differentiable on that region. Remarkably, it can be shown that equation (32.8) can also be used to verify complex differentiability. More precisely, the following theorem can be verified using tools developed in a typical course in complex analysis.

Theorem 32.13

A function f is complex differentiable on an open region if and only if

$$\frac{\partial}{\partial y} f(x+iy) = i\frac{\partial}{\partial x} f(x+iy)$$

at every point $z = x+iy$ in the region. Moreover, in any open region on which f is complex differentiable,

$$f'(z) = \frac{d}{dz} f(z) = \frac{\partial}{\partial x} f(x+iy) \quad .$$

[4] The two equations you get by splitting equation (32.8) into its real and imaginary parts are the famous *Cauchy-Riemann equations*.

Differentiability of an Analytic Function

In the subsection starting on page 30–12 of section 30.2, we discussed differentiating power series and analytic functions when the variable is real. That discussion remains true if we replace the real variable x with the complex variable z and use the complex derivative. In particular, we have

Theorem 32.14 (differentiation of power series)
Suppose f is a function given by a power series,

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k \qquad \text{for} \quad |z - z_0| < R$$

for some $R > 0$. Then, for any positive integer n, the n^{th} derivative of f exists. Moreover,

$$f^{(n)}(x) = \sum_{k=n}^{\infty} k(k-1)(k-2)\cdots(k-n+1)\, a_k (z - z_0)^{k-n} \qquad \text{for} \quad |z - z_0| < R \quad .$$

As an immediate corollary, we have:

Corollary 32.15
Let f be analytic at z_0 with power series representation

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k \qquad \text{whenever} \quad |z - z_0| < R \quad .$$

Then f is infinitely complex differentiable on the disk of all z with $|z - z_0| < R$. Moreover

$$a_k = \frac{f^{(k)}(z_0)}{k!} \qquad \text{for} \quad k = 0, 1, 2, \ldots \quad .$$

Complex Differentiability and Analyticity

Despite the similarity between complex differentiation and real-variable differentiation, complex differentiability is a much stronger condition on functions than is real-variable differentiability. The next theorem illustrates this.

Theorem 32.16
Assume $f(z)$ is complex differentiable in some open region $\mathcal{R}$. Then f is analytic at each point z_0 in $\mathcal{R}$, and is given by its Taylor series

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!} (z - z_0)^k \qquad \text{whenever} \quad |z - z_0| < R$$

where R is the radius of any open disk centered at z_0 and contained in region $\mathcal{R}$.

This remarkable theorem tells us that complex differentiability on an open region automatically implies analyticity on that region, and tells us the region over which a function's Taylor series converges and equals the function. Proving this theorem is beyond this text. It is, in fact, a summary of results normally derived over the course of many chapters of a typical text in complex variables.

Keep in mind that we already saw that analyticity implied complex differentiability (corollary 32.15). So as immediate corollaries to the above, we have:

Corollary 32.17
A function f is analytic at every point in an open region of the complex plane if and only if it is complex differentiable at every point in that region.

Corollary 32.18
Assume F is a function analytic at z_0 with corresponding power series $\sum_{k=0}^{\infty} f_k(z-z_0)^k$, and let R be either some positive value or $+\infty$. Then

$$F(z) = \sum_{k=0}^{\infty} f_k(z-z_0)^k \qquad \text{whenever} \quad |z-z_0| < R$$

if and only if F is analytic at every complex point z satisfying

$$|z-z_0| < R \quad .$$

The second corollary is especially of interest to us because it is the same as lemma 32.1 on page 32–3, which we used extensively in this chapter. And the other lemma that we used, lemma 32.2 on page 32–3? Well, using, in order,

1. corollary 30.14 on page 30–22 on quotients of analytic functions,

2. theorem 32.12 on page 32–20 on the differentiation of products and quotients

and

3. corollary 32.17, above

you can easily verify the following (which is the same as lemma 32.2):

Corollary 32.19
Assume $F(z)$ and $A(z)$ are two functions analytic at a point z_0 . Then the quotient $^F/_A$ is also analytic at z_0 if and only if

$$\lim_{z \to z_0} \frac{F(z)}{A(x)}$$

is finite.

The details are left to you.

?►Exercise 32.4: *Prove corollary 32.19.*

32.8 Appendix: The "Closest Singular Point"

Here we want to answer a subtle question: Is it possible to have a first- or second-order linear homogeneous differential equation whose singular points are arranged in such a manner that none of them is the closest to some given ordinary point?

For example, could there be a differential equation having $z_0 = 0$ as an ordinary point, but whose singular points form an infinite sequence

$$z_1,\ z_2,\ z_3,\ \ldots \qquad \text{with} \quad |z_k| \;=\; 1 + \frac{1}{k} \quad ,$$

possibly located in the complex plane so that they "spiral around" the circle of radius 1 about $z_0 = 0$ without converging to some single point? Each of these singular points is closer to $z_0 = 0$ than the previous ones in the sequence, so not one of them can be called "a closest singular point".

Lemma 32.6 on page 32–6 claims that this situation can*not* happen. Let us see why we should believe this lemma.

The Problem and Fundamental Theorem

We are assuming that we have some first- or second-order linear homogeneous differential equation having singular points. We also assume z_0 is not one of these singular points — it is an ordinary point. Our goal is to show that

there is a singular point z_s such that no other singular point is closer to z_0.

If we can confirm such a z_s exists, then we've shown that the answer to this section's opening question is *No* (and proven lemma 32.6).

We start our search for this z_s by rewriting our differential equation in reduced form

$$y' \;+\; P(x)y \;=\; 0 \qquad \text{or} \qquad y'' \;+\; P(x)y' \;+\; Q(x)y \;=\; 0$$

and recalling that a point z is an ordinary point for our differential equation if and only if the coefficient(s) (P for the first-order equation, and both P and Q for the second-order equation) are all analytic at z (see lemmas 32.7 and 32.8). Consequently,

1. z_s is a closest singular point to z_0 for the first-order differential equation if and only if z_s is a point closest to z_0 at which P is not analytic,

and

2. z_s is a closest singular point to z_0 for the second-order differential equation if and only if z_s is a point closest to z_0 at which either P or Q is not analytic.

Either way, our problem of verifying the existence of a singular point z_s "closest to z_0" is reduced to the problem of verifying the existence of a point z_s "closest to z_0" at which a given function F is not analytic while still being analytic at z_0. That is, to prove lemma 32.6, it will suffice to prove the following:

Theorem 32.20
Let F be a function that is analytic at some, but not all, points in the complex plane, and let z_0 be one of the points at which F is analytic. Then there is a positive value R_0 and a point z_s in the complex plane such that all the following hold:

1. $R_0 = |z_s - z_0|$.
2. *F is not analytic at z_s .*
3. *F is analytic at every z with $|z - z_0| < R_0$.*

The point z_s in the above theorem is a point closest to z_0 at which F is not analytic. There may be other points the same distance from z_0 at which F is not analytic, but the last statement in the theorem tells us that there is no point closer to z_0 than z_s at which F is not analytic.

Verifying Theorem 32.20
The Radius of Analyticity Function

Our proof of theorem 32.20 will rely on properties of the *radius of analyticity function R_A for F*, which we define at each point z in the complex plane as follows:

- If F is not analytic at z, then $R_A(z) = 0$.
- If F is analytic at z, then $R_A(z)$ is the largest value of R such that

$$F \text{ is analytic on the open disk of radius } R \text{ about } z \quad . \tag{32.9}$$

(To see that this “largest value of R” exists when f is analytic at z, first note that the set of all positive values of R for which (32.9) holds forms an interval with 0 as the lower endpoint. Since we are assuming there are points at which F is not analytic, this interval must be finite, and, hence, has a finite upper endpoint. That endpoint is $R_A(z)$.)[5]

The properties of this function that will be used in our proof of theorem 32.20 are summarized in the following lemmas.

Lemma 32.21
Let R_A be the radius of analyticity function corresponding to a function F analytic at some, but not all, points of the complex plane, and let z_0 be a point at which F is analytic. Then:

1. *If $|\zeta - z| < R_A(z)$, then F is analytic at ζ .*
2. *If F is not analytic at a point ζ , then $|\zeta - z| \geq R_A(z)$.*
3. *If $R > R_A(z)$ then there is a point in the open disk about z of radius R at which F is not analytic.*

Lemma 32.22
Let F be a function which is analytic at some, but not all, points of the complex plane, and let R_A be the radius of analyticity function corresponding to F . Then, for each complex point z ,

$$F \text{ is analytic at } z \quad \iff \quad R_A(z) > 0 \quad .$$

Equivalently,

$$F \text{ is not analytic at } z \quad \iff \quad R_A(z) = 0 \quad .$$

[5] In practice, $R_A(z)$ is usually the radius of convergence R for the Taylor series for F about z . In theory, though, one can define F to not equal its Taylor series at some points in the disk of radius R about z . $R_A(z)$ is then the radius of the largest open disk about z on which F is given by its Taylor series about z .

Lemma 32.23

If F is a function which is analytic at some, but not all, points of the complex plane, then R_A, the radius of analyticity function corresponding to F, is a continuous function on the complex plane.

The claims in the first lemma follow immediately from the definition of R_A; so let us concentrate on proving the other two lemmas.

PROOF (lemma 32.22): First of all, by definition

$$F \text{ is not analytic at } z \quad\Longrightarrow\quad R_A(z) = 0 \quad .$$

Hence, we also have

$$R_A(z) > 0 \quad\Longrightarrow\quad F \text{ is analytic at } z \quad .$$

On the other hand, if F is analytic at z, then there is a power series $\sum_{k=0}^{\infty} a_k(\zeta - z)^k$ and a $R > 0$ such that

$$F(\zeta) = \sum_{k=0}^{\infty} a_k(\zeta - z)^k \qquad \text{whenever} \quad |\zeta - z| < R \quad .$$

Corollary 32.18 immediately tells us that F is analytic on the open disk of radius R about z. Since $R_A(z)$ is the largest such R, $R \leq R_A(z)$. And since $0 < R$, we now have

$$F \text{ is analytic at } z \quad\Longrightarrow\quad R_A(z) > 0 \quad .$$

This also means

$$R_A = 0 \quad\Longrightarrow\quad F \text{ is not analytic at } z \quad .$$

Combining all the implications just listed yields the claims in the lemma. ∎

PROOF (lemma 32.23): To verify the continuity of R_A, we need to show that

$$\lim_{z \to z_1} R_A(z) = R_A(z_1) \qquad \text{for each complex value } z_1 \quad .$$

There are two cases: The easy case with F not being analytic at z_1, and the less-easy case with F being analytic at z_1. For the second case, we will use pictures.

Consider the first case, where F is not analytic at z_1 (hence, $R_A(z_1) = 0$). Then, as noted in lemma 32.21,

$$0 \leq R_A(z) \leq |z_1 - z| \qquad \text{for any } z \text{ in } \mathbb{C} \quad .$$

Taking limits, we see that

$$0 \leq \lim_{z \to z_1} R_A(z) \leq \lim_{z \to z_1} |z_1 - z| = 0 \quad ,$$

which, since $R_A(z_1) = 0$, gives us

$$\lim_{z \to z_1} R_A(z) = R_A(z_1) \quad .$$

Next, assume F is analytic at z_1 (so that $R_A(z_1) > 0$), and let z be a point "close" to z_1. For notational convenience, let

$$R_1 = R_A(z_1)$$

and let $\mathcal{D}_1$ be the open disk centered at z_1 of radius R_1, as sketched in figure 32.1. Note that, by the definition of R_1,

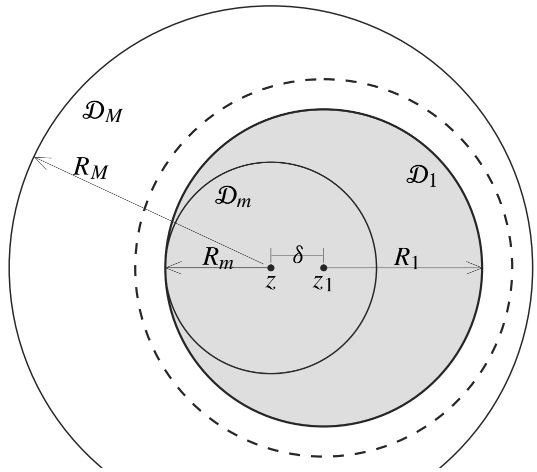

Figure 32.1: Limits on the radius of analyticity at z based on the radius of analyticity R_1 at z_1. Here $\delta = |z - z_1|$ $R_M = R_1 + 2\delta$ and $R_m = R_1 - \delta$.

1. F is analytic at every point in $\mathcal{D}_1$, but
2. any open disk centered at z_1 which is larger than $\mathcal{D}_1$ (such as the one in figure 32.1 enclosed by the dashed-line circle) must contain a point at which F is not analytic.

Because we are just interested in limits as $z \to z_1$, we can assume z is close enough to z_1 that $|z - z_1| < R_1$. Let $\mathcal{D}_m$ and $\mathcal{D}_M$ be the open disks about z with radii

$$R_m = R_1 - |z - z_1| \qquad \text{and} \qquad R_M = R_1 + 2|z - z_1| \quad ,$$

as also illustrated in figure 32.1. Now since $\mathcal{D}_m$ is contained in $\mathcal{D}_1$, F is analytic at every point in $\mathcal{D}_m$. Hence,

$$R_m \leq R_A(z) \quad .$$

On the other hand, inside $\mathcal{D}_M$ is another open disk that we had already noted contains a point at which F is not analytic. So F is not analytic at every point in $\mathcal{D}_M$. Thus,

$$R_A(z) < R_M \quad .$$

Combining the two inequalities above (and recalling what the notation means) gives us

$$R_A(z_1) - |z - z_1| \leq R_A(z) \leq R_A(z_1) + 2|z - z_1|$$

which, after letting $z \to z_1$, becomes

$$R_A(z_1) - 0 \leq \lim_{z \to z_1} R_A(z) \leq R_A(z_1) + 2 \cdot 0 \quad ,$$

clearly implying that

$$\lim_{z \to z_1} R_A(z) = R_A(z_1)$$

∎

Proof of Theorem 32.20

Remember: F is a function analytic at some, but not all, points in the complex plane, and z_0 is a point at which f is analytic.

Let

$$R_0 = R_A(z_0) \quad .$$

Because of the definition and properties of R_A, and the analyticity of F at z_0, we automatically have that $R_0 > 0$ and that F is analytic at every z with $|z - z_0| < R_0$. All that remains to proving our theorem is to show that there is a z_s at which F is not analytic and which satisfies $|z_s - z_0| = R_0$.

Now consider the possible values of $R_A(z)$ on the circle $|z - z_0| = R_0$. Since R_A is continuous, it must have a minimum value ρ at some point z_s in this circle. Since this is a minimum value for R_A on the circle, we then have

$$0 \geq \rho \geq R_A(z) \qquad \text{whenever} \quad |z - z_0| = R_0 \quad .$$

If $\rho > 0$, then the above implies that F is analytic at every point closer than ρ to the circle $|z - z_0| = R_0$, as well at every point inside the disk enclosed by this circle. That is, F_A is analytic at every z on the open disk about z_0 of radius $R_0 + \rho$. And by the definition of R_A and R_0 and ρ, we must then have

$$R_A(z_0) = R_0 \leq R_0 + \rho \leq R_A(z_0) \quad ,$$

which is only possible if $\rho = 0$. But $\rho = R_A(z_s)$. So $R_A(z_s) = 0$, which, as lemma 32.22 tells us, means that F is not analytic at z_s. ∎

32.9 Appendix: Singular Points and the Radius of Convergence for Power Series Solutions

Our goal in this section is to prove theorem 32.11, which directly relates the radius of convergence for a power series solution for a given differential equation to a singular point for that differential equation. To do this, we must first expand on some of our discussion from the last few sections.

Analytic Continuation

Analytic continuation is any procedure that "continues" an analytic function defined on one region so that it becomes defined on a larger region. Perhaps it would be better to call it "analytic extension" because what we are really doing is extending the domain of our original function by creating an analytic function with a larger domain that equals the original function over the original domain.

We will "analytically extend" a power series solution to our differential equation on one disk to a solution on a larger disk using Taylor series. And to justify this, we will use the following theorem:

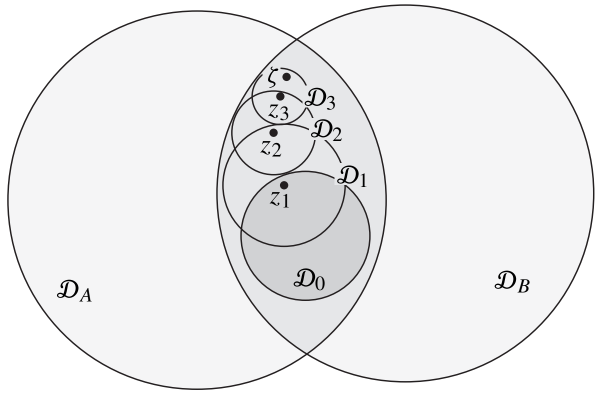

Figure 32.2: Disks in the complex plane for theorem 32.24 and its proof.

Theorem 32.24

Let $\mathcal{D}_A$ and $\mathcal{D}_B$ be two open disks in the complex plane that intersect each other, and let f_A be a function analytic on $\mathcal{D}_A$ and f_B a function analytic on $\mathcal{D}_B$. Assume further that there is an open disk $\mathcal{D}_0$ contained in both $\mathcal{D}_A$ and $\mathcal{D}_B$ (see figure 32.2), and that

$$f_A(z) = f_B(z) \qquad \text{for every} \quad z \in \mathcal{D}_0 \quad .$$

Then

$$f_A(z) = f_B(z) \qquad \text{for every} \quad z \in \mathcal{D}_A \cap \mathcal{D}_B \quad .$$

Think of f_A as being the original function defined on $\mathcal{D}_A$, and f_B as some other analytic function that we constructed on $\mathcal{D}_B$ to match f_A on $\mathcal{D}_0$. This theorem tells us that we can define a "new" function f on $\mathcal{D}_A \cup \mathcal{D}_B$ by

$$f(z) = \begin{cases} f_A(z) & \text{if} \quad z \text{ is in } \mathcal{D}_A \\ f_B(z) & \text{if} \quad z \text{ is in } \mathcal{D}_B \end{cases} \quad .$$

On the intersection, f is given both by f_A and f_B, but that is okay because the theorem assures us that f_A and f_B are the same on that intersection. And since f_A and f_B are, respectively, analytic at every point in $\mathcal{D}_A$ and $\mathcal{D}_B$, it follows that f is analytic on the union of $\mathcal{D}_A$ and $\mathcal{D}_B$, and satisfies

$$f(z) = f_A(z) \qquad \text{for each } z \text{ in } \mathcal{D}_A \quad .$$

That is, f is an "analytic extension" of f_A from the domain $\mathcal{D}_A$ to the domain $\mathcal{D}_A \cup \mathcal{D}_B$.

The proof of the above theorem is not difficult, and is somewhat instructive.

PROOF (theorem 32.24): We need to show that $f_A(\zeta) = f_B(\zeta)$ for every ζ in $\mathcal{D}_A \cap \mathcal{D}_B$. So let ζ be any point in $\mathcal{D}_A \cap \mathcal{D}_B$.

If $\zeta \in \mathcal{D}_0$ then, by our assumptions, we automatically have $f_A(\zeta) = f_B(\zeta)$.

On the other hand, if ζ in not in $\mathcal{D}_0$, then, as illustrated in figure 32.2, we can clearly find a finite sequence of open disks $\mathcal{D}_1$, $\mathcal{D}_2$, … and $\mathcal{D}_M$ with respective centers z_1, z_2, … and z_M such that

1. each z_k is also in $\mathcal{D}_{k-1}$,

2. each $\mathcal{D}_k$ is in $\mathcal{D}_A \cap \mathcal{D}_B$, and

3. the last disk, $\mathcal{D}_M$, contains ζ .

Now, because f_A and f_B are the same on $\mathcal{D}_0$, so are all their derivatives. Consequently, the Taylor series for f_A and f_B about the point z_1 in $\mathcal{D}_0$ will be the same. And since $\mathcal{D}_1$ is a disk centered at z_1 and contained in both $\mathcal{D}_A$ and $\mathcal{D}_B$, we have

$$\begin{aligned} f_A(z) &= \sum_{k=0}^{\infty} \frac{f_A{}^{(k)}(z_1)}{k!}(z - z_1)^k \\ &= \sum_{k=0}^{\infty} \frac{f_B{}^{(k)}(z_1)}{k!}(z - z_1)^k = f_B(z) \qquad \text{for every } z \text{ in } \mathcal{D}_1 \quad . \end{aligned}$$

Repeating these arguments using the Taylor series for f_A and f_B at the points z_2 , z_3 , and so on, we eventually get

$$f_A(z) = f_B(z) \qquad \text{for every } z \text{ in } \mathcal{D}_M \quad .$$

In particular then,

$$f_A(\zeta) = f_B(\zeta) \quad ,$$

just as we wished to show. ∎

Ordinary and Singular Points for Power-Series Functions

It will help if we expand our notions of ordinary and singular points to any function given by a power series,

$$f(z) = \sum_{k=0}^{\infty} c_k(z - z_0)^k \qquad \text{for} \quad |z - z_0| < R \quad ,$$

assuming, of course, that $R > 0$. For convenience, let $\mathcal{D}$ be the open disk about z_0 of radius R . Then for each z_1 either in $\mathcal{D}$ or on its boundary, we will say:

1. z_1 an *ordinary point* for f if and only if there is a function f_1 analytic on a disk $\mathcal{D}_1$ of positive radius about z_1 and which equals f on the region where $\mathcal{D}$ and $\mathcal{D}_1$ overlap.

2. z_1 an *singular point* for f if and only if it is not an ordinary point for f .

Do note that, theorem 32.16 on page 32–21 assures us that every point in $\mathcal{D}$ is an ordinary point for f . So the only singular points must be on the boundary. And the next theorem tells us that there must be a singular point on the boundary of $\mathcal{D}$ when R is finite and the radius of convergence for the above power series.

Theorem 32.25
Let R be a positive finite number, and assume it is the radius of convergence for

$$f(z) = \sum_{k=0}^{\infty} c_k(z - z_0)^k \quad .$$

Then f must have a singular point on the circle $|z - z_0| = R$.

PROOF: For convenience, let $\mathcal{D}$ be the open disk of radius R about z_0,

$$\mathcal{D} = \{z : |z - z_0| < R\} \quad ,$$

and let $\overline{\mathcal{D}}$ be the union of $\mathcal{D}$ and its boundary,

$$\overline{\mathcal{D}} = \{z : |z - z_0| \leq R\} \quad .$$

Now, let's define a function R_C on $\overline{\mathcal{D}}$ as follows:

1. For each ordinary point ζ, there is a function f_1 analytic on a disk $\mathcal{D}_1$ of positive radius about ζ and which equals f on the region where $\mathcal{D}$ and $\mathcal{D}_1$ overlap. Since f_1 is analytic, it can be given by a power series about ζ. Let $R_C(\zeta)$ be the radius of convergence for that power series.

2. For each singular point ζ, let $R_C(\zeta) = 0$.

By definition, $R_C(\zeta) \geq 0$ for each ζ in $\overline{\mathcal{D}}$. We should also note that $R_C(\zeta)$ cannot be infinite for any ζ in $\overline{\mathcal{D}}$ because there would then be a function f_1 analytic on all of the complex plane and equaling f on the disk $\mathcal{D}$. And since f_1 is analytic everywhere, theorem 32.16 assures us that the radius of convergence R_1 of the Taylor series of f_1 about z_0 would be infinite. But that Taylor series for f_1 about z_0 would have to be the same as the Taylor series for f about z_0 since $f = f_1$ on $\mathcal{D}$. And that, in turn, would mean the two power series have the same radius of convergence, giving us

$$\infty = R_1 = R < \infty \quad ,$$

which is impossible. Thus, it is not possible for $R_C(\zeta)$ to be infinite at any point ζ in $\overline{\mathcal{D}}$. In other words, R_C is a well-defined function on $\mathcal{D}$ with

$$0 \leq R_C(z) < \infty \qquad \text{for each } z \text{ in } \overline{\mathcal{D}} \quad .$$

The function R_C is very similar to the "radius of analyticity function" R_A discussed in section 32.8, and, using arguments very similar to those used in that section for R_A, you can verify that

1. $R_C(z) > 0$ if and only if z is an ordinary point for the differential equation.

2. $R_C(z) = 0$ if and only if z is a singular point for the differential equation.

3. R_C is a continuous function on the circle $|z - z_0| = R$.

4. $R_C(z)$ has a minimum value ρ at some point z_s on the circle $|z - z_0| = R$.

Now, if we can show $\rho = 0$, then the above tells us that the corresponding point z_s is a singular point for the given power series, and our theorem is proven. And to show that, it will suffice to show that $\rho > 0$ is impossible.

So, for the moment, assume $\rho > 0$. Then (as illustrated in figure 32.3) we can choose a finite sequence of points ζ_1, ζ_2, ..., and ζ_N about the circle $|z - z_0| = R$ such that $|\zeta_N - \zeta_1| < \rho$ and

$$|\zeta_{n+1} - \zeta_n| < \rho \qquad \text{for} \quad n = 1, 2, 3, \ldots, n \quad .$$

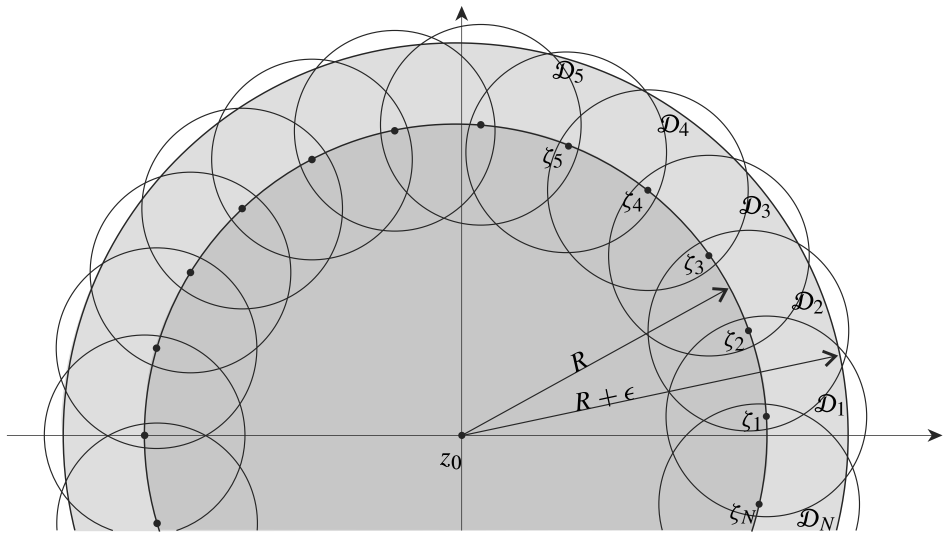

Figure 32.3: Disks for the proof of theorem 32.25. The darker disk is the disk of radius R on which y is originally defined.

For each ζ_n, let $\mathcal{D}_n$ be the open disk of radius ρ about ζ_n, and observe that, by basic geometry, the union of all the disks $\mathcal{D}_1$, $\mathcal{D}_2$, ..., and $\mathcal{D}_N$ contains not just the boundary of our original disk $\mathcal{D}$ but the annulus of all z satisfying

$$R \leq |z - z_0| \leq R + \epsilon$$

for some positive value ϵ.

By our choice of ζ_n's and $\mathcal{D}$'s, we know that, for each integer n from 1 to N, there is a power series function

$$f_n(z) = \sum_{k=0}^{\infty} c_{n,k}(z - \zeta_n)^k$$

defined on all of $\mathcal{D}_n$ and which equals our original function f in the overlap of $\mathcal{D}$ and $\mathcal{D}_n$. Repeated use of theorem 32.24 then shows that any two functions from the set

$$\{f, f_1, f_2, \ldots, f_N\}$$

equal each other wherever both are defined. This allows us to define a "new" analytic function F on the union of all of our disks via

$$F(z) = \begin{cases} f(z) & \text{if } |z - z_0| < R \\ f_n(z) & \text{if } z \in \mathcal{D}_n \quad \text{for } n = 1, 2, \ldots, N \end{cases} .$$

Now, because the union of all of the disks contains the disk of radius $R + \epsilon$ about 0, theorem 32.16 on page 32–21 on the radius of convergence for Taylor series assures us that the Taylor series for F about z_0 must have a radius of convergence of at least $R + \epsilon$. But, $f(z) = F(z)$ when $|z| < R$. So f and F have the same Taylor series at z_0, and, hence, these two power series share the same radius of convergence.

That is, if $\rho > 0$, then

$$\begin{aligned} R &= \text{radius of convergence for the power series of } f \text{ about } z_0 \\ &= \text{radius of convergence for the power series of } F \text{ about } z_0 > R \quad , \end{aligned}$$

which is clearly impossible. So, it is not possible to have $\rho > 0$. ∎

Complex Power Series Solutions

Throughout most of these chapters, we've been tacitly assuming that the derivatives in our differential equations are derivatives with respect to a real variable x,

$$y' = \frac{dy}{dx} \quad \text{and} \quad y'' = \frac{d^2y}{dx^2} \quad ,$$

as in elementary calculus. In fact, we can also have derivatives with respect to the complex variable z,

$$y' = \frac{dy}{dz} \quad \text{and} \quad y'' = \frac{d^2y}{dz^2} \quad .$$

as described on page 32–19). Remember that, computationally, differentiation with respect to z is completely analogous to the differentiation with respect to x learned in basic calculus. In particular, if

$$y(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k \quad \text{for} \quad |z - z_0| < R$$

for some point z_0 in the complex plane and some $R > 0$, then

$$\begin{aligned} y'(z) &= \frac{d}{dz} \sum_{k=0}^{\infty} a_k (z - z_0)^k \\ &= \sum_{k=0}^{\infty} \frac{d}{dz}\left[a_k (z - z_0)^k\right] = \sum_{k=1}^{\infty} k a_k (z - z_0)^{k-1} \quad \text{for} \quad |z - z_0| < R \quad . \end{aligned}$$

Consequently, all of our computations in chapters 31 and 32 can be carried out using the complex variable z instead of the real variable x, and using a point z_0 in the complex plane instead of a point x_0 on the real line. In particular, we have the following complex-variable analogs of theorems 32.9 and 32.10:

Theorem 32.26 (first-order series solutions)

Suppose z_0 is an ordinary point for a first-order homogeneous differential equation whose reduced form is

$$y' + Py = 0 \quad .$$

Then P has a power series representation

$$P(z) = \sum_{k=0}^{\infty} p_k (z - z_0)^k \quad \text{for} \quad |z - z_0| < R$$

where R is the radius of analyticity about z_0 for this differential equation.

Moreover, a general solution to the differential equation is given by

$$y(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k \quad \text{for} \quad |z - z_0| < R$$

where a_0 *is arbitrary, and the other* a_k*'s satisfy the recursion formula*

$$a_k \;=\; -\frac{1}{k}\sum_{j=0}^{k-1} a_j p_{k-1-j} \quad . \tag{32.10}$$

Theorem 32.27 (second-order series solutions)

Suppose z_0 *is an ordinary point for a second-order homogeneous differential equation whose reduced form is*

$$\frac{d^2y}{dz^2} \;+\; P\frac{dy}{dz} \;+\; Qy \;=\; 0 \quad .$$

Then P *and* Q *have power series representations*

$$P(z) \;=\; \sum_{k=0}^{\infty} p_k (z - z_0)^k \qquad \text{for} \quad |z - z_0| < R$$

and

$$Q(z) \;=\; \sum_{k=0}^{\infty} q_k (z - z_0)^k \qquad \text{for} \quad |z - z_0| < R$$

where R *is the radius of analyticity about* z_0 *for this differential equation.*

Moreover, a general solution to the differential equation is given by

$$y(z) \;=\; \sum_{k=0}^{\infty} a_k (z - z_0)^k \qquad \text{for} \quad |z - z_0| < R$$

where a_0 *and* a_1 *are arbitrary, and the other* a_k*'s satisfy the recursion formula*

$$a_k \;=\; -\frac{1}{k(k-1)}\sum_{j=0}^{k-2}\left[(j+1)a_{j+1}p_{k-2-j} + a_j q_{k-2-j}\right] \quad . \tag{32.11}$$

By the way, it should also be noted that, if

$$y(x) \;=\; \sum_{k=0}^{\infty} c_k (x - x_0)^k$$

is a solution to the real-variable differential equation for $|x - x_0| < R$, then

$$y(z) \;=\; \sum_{k=0}^{\infty} c_k (z - x_0)^k$$

is a solution to the corresponding complex-variable differential equation for $|z - x_0| < R$. This follows immediately from the last theorem and the relation between dy/dx and dy/dz (see the discussion of the complex derivative in section 32.7).

Singular Points of Differential Equations and Solutions

So, let's suppose we have a power series solution

$$y(z) = \sum_{k=0}^{\infty} c_k (z - z_0)^k \qquad \text{for} \quad |z - x_0| < R$$

to some first- or second-order linear homogenous differential equation, and, as before, let $\mathcal{D}$ and $\overline{\mathcal{D}}$ be, respectively, the open and closed disks of radius R about z_0,

$$\mathcal{D} = \{z : |z - z_0| < R\} \qquad \text{and} \qquad \overline{\mathcal{D}} = \{z : |z - z_0| \leq R\} \quad .$$

Now consider any single point z_1 in $\overline{\mathcal{D}}$. If z_1 is an ordinary point for the given differential equation, then there is a disk $\mathcal{D}_1$ of some positive radius R_1 about z_1 such that, on that disk, general solutions to the differential equation exist and are given by power series about z_1. Using the material already developed in this section, it's easy to verify that, in particular, the above power series solution y is given by a power series about z_1 at least on the overlap of disks $\mathcal{D}$ and $\mathcal{D}_1$. And that means z_1 is also an ordinary point for the above power series solution. And, of course, this also means that a point z_1 in $\overline{\mathcal{D}}$ cannot be both an ordinary point for the differential equation and a singular point for y.

To summarize:

Lemma 32.28
Let R be a positive finite number, and assume that, on the disk $|z - z_0| < R$,

$$y(z) = \sum_{k=0}^{\infty} c_k (z - z_0)^k$$

is a power series solution to some first- or second-order linear homogeneous differential equation. Then, for each point ζ in the closed disk given by $|z - z_0| \leq R$:

1. *If ζ is an ordinary point for the differential equation, then it is an ordinary point for the above power series solution.*
2. *If ζ is a singular point for the above power series solution, then it is a singular point for the differential equation.*

Combining the last lemma with theorem 32.25 on singular points for power series functions, we get the main result of this appendix:

Theorem 32.29
Let

$$y(z) = \sum_{k=0}^{\infty} c_k (z - z_0)^k \qquad \text{for} \quad |z - z_0| < R$$

be a power series solution for some first- or second-order homogeneous linear differential equation. Assume, further, that R is finite and is the radius of convergence for the above power series. Then there is a point z_s with $|z_s - z_0| = R$ which is a singular point for both the above power series solution and the given differential equation.

Theorem 32.11 now follows as a corollary of the last theorem.

Additional Exercises

32.5 a. *Using the fact that*

$$e^{x+iy} = e^x [\cos(y) + i \sin(y)] \quad ,$$

show that e^z *can never equal zero for any* z *in the complex plane.*

b. *In chapter 16 we saw that*

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i} \qquad \text{and} \qquad \cos(z) = \frac{e^{iz} + e^{-iz}}{2} \quad ,$$

at least when z *was a real value (see page 398). In fact, these formulas (along with the definition of the complex exponential) can define the sine and cosine functions at all values of* z *, real and complex. Using these formulas*

i. *Verify that*

$$\sin(x + iy) = \frac{1}{2}\left\{\left[e^y + e^{-y}\right]\sin(x) + i\left[e^y - e^{-y}\right]\cos(x)\right\}$$

and

$$\cos(x + iy) = \frac{1}{2}\left\{\left[e^y + e^{-y}\right]\cos(x) - i\left[e^y - e^{-y}\right]\sin(x)\right\} \quad .$$

ii. *Using the above formulas, verify that*

$$\sin(z) = 0 \iff z = n\pi \qquad \text{with} \quad n = 0, \pm 1, \pm 2, \ldots$$

and

$$\cos(z) = 0 \iff z = \left[n + \tfrac{1}{2}\right]\pi \qquad \text{with} \quad n = 0, \pm 1, \pm 2, \ldots \quad .$$

c. *The hyperbolic sine and cosine are defined on the complex plane by*

$$\sinh(z) = \frac{e^z - e^{-z}}{2} \qquad \text{and} \qquad \cosh(z) = \frac{e^z + e^{-z}}{2} \quad .$$

Show that

$$\sinh(z) = 0 \iff z = in\pi \qquad \text{with} \quad n = 0, \pm 1, \pm 2, \ldots$$

and

$$\cosh(z) = 0 \iff z = i\left[n + \tfrac{1}{2}\right]\pi \qquad \text{with} \quad n = 0, \pm 1, \pm 2, \ldots \quad .$$

32.6. *Find all singular points for each of the following differential equations. You may have to use results from the previous problem. You may even have to expand on some of those results. And you may certainly need to rearrange a few equations.*

a. $y' - e^x y = 0$

b. $y' - \tan(x)y = 0$

c. $\sin(\pi x)\, y'' + x^2 y' - e^x y = 0$

d. $\sinh(x)\, y'' + x^2 y' - e^x y = 0$

e. $\sinh(x)\,y'' + x^2 y' - \sin(x)\,y = 0$

f. $e^{3x} y'' + \sin(x)\,y' + \dfrac{2}{(x^2+4)} y = 0$

g. $y'' - \dfrac{1+e^x}{(1-e^x)} y$

h. $\left[x^2 - 4\right] y'' + \left[x^2 + x - 6\right] y = 0$

i. $xy'' + \left[1 - e^x\right] y = 0$

j. $\sin\left(\pi x^2\right) y'' + x^2 y = 0$

32.7. *Using recursion formula (32.3) on page 32–10, find the* 4^{th}*-degree partial sum of the general power series solution for each of the following about the given point* x_0 *. Also state the interval over which you can be sure the full power series solution is valid.*

a. $y' - e^x y = 0$ *with* $x_0 = 0$

b. $y' + e^{2x} y = 0$ *with* $x_0 = 0$

c. $y' + \cos(x)\,y = 0$ *with* $x_0 = 0$

d. $y' + \ln|x|\,y = 0$ *with* $x_0 = 1$

32.8. *Using the recursion formula (32.4) on page 32–10, find the* 4^{th}*-degree partial sum of the general power series solution for each of the following about the given point* x_0 *. Also state the interval over which you can be sure the full power series solution is valid.*

a. $y'' - e^x y = 0$ *with* $x_0 = 0$

b. $y'' + 3xy' - e^x y = 0$ *with* $x_0 = 0$

c. $xy'' + \sin(x)\,y = 0$ *with* $x_0 = 0$

d. $y'' + \ln|x|\,y = 0$ *with* $x_0 = 1$

e. $\sqrt{x}\,y'' - y = 0$ *with* $x_0 = 1$

f. $y'' + \left[1 + 2x + 6x^2\right] y' + [2 + 12x]\,y = 0$ *with* $x_0 = 0$

32.9 **a.** *Using your favorite computer mathematics package (e.g., Maple or Mathematica), along with recursion formula (32.3) on page 32–10, write a program/worksheet that will find the* N^{th} *partial sum of the power series solution about* x_0 *to*

$$y' + P(x)y = 0$$

for any given positive integer N *, point* x_0 *and function* $P(x)$ *analytic at* x_0 *. Finding the appropriate partial sum of the corresponding power series for* P *should be part of the program/worksheet (see exercise 30.8 on page 30–26). Be sure to write your program/worksheet so that* N *,* x_0 *and* P *are easily changed.*

b. *Use your program/worksheet to find the* N^{th}*-degree partial sum of the general power series solution about* x_0 *for each of the following differential equations and choices for* N *and* x_0 *.*

i. $y' - e^x y = 0$ *with* $x_0 = 0$ *and* $N = 10$

ii. $y' + \sqrt{x^2+1}\,y = 0$ *with* $x_0 = 0$ *and* $N = 8$

iii. $\cos(x)y' + y = 0$ *with* $x_0 = 0$ *and* $N = 8$

iv. $y' + \sqrt{2x^2+1}\,y = 0$ *with* $x_0 = 2$ *and* $N = 5$

32.10 a. *Using your favorite computer mathematics package (e.g., Maple or Mathematica), along with recursion formula (32.4) on page 32–10, write a program/worksheet that will find the N^{th} partial sum of the power series solution about x_0 to*

$$y'' + P(x)y' + Q(x)y = 0$$

for any given positive integer N, point x_0, and functions $P(x)$ and $Q(x)$ analytic at x_0. Finding the appropriate partial sum of the corresponding power series for P and Q should be part of the program/worksheet (see exercise 30.8 on page 30–26). Be sure to write your program/worksheet so that N, x_0, P and Q are easily changed.

b. *Use your program/worksheet to find the N^{th}-degree partial sum of the general power series solution about x_0 for each of the following differential equations and choices for N and x_0.*

i. $y'' - e^x y = 0$ *with* $x_0 = 0$ *and* $N = 8$

ii. $y'' + \cos(x)y = 0$ *with* $x_0 = 0$ *and* $N = 10$

iii. $y'' + \sin(x)y' + \cos(x)y = 0$ *with* $x_0 = 0$ *and* $N = 7$

iv. $\sqrt{x}\,y'' + y' + xy = 0$ *with* $x_0 = 1$ *and* $N = 5$

32.11. *In this problem, we will compare two ways of finding the general power series solution $\sum_{k=0}^{\infty} a_k x^k$ to*

$$(3-x)y' - y = 0 \quad .$$

a. *Using the algebraic method from section 31.2:*

i. *Find the corresponding recursion formula for the a_k's.*

ii. *Find the 6^{th}-degree partial sum for the general power series method.*

b. *Observe that the reduced form of our differential equation is*

$$y' - \frac{1}{3-x}y = 0 \quad ,$$

and verify that

$$\frac{1}{3-x} = \sum_{k=0}^{\infty} \frac{1}{3^{k+1}} x^k \quad .$$

Then, using theorem 32.9:

i. *Find the corresponding recursion formula for the a_k's.*

ii. *Find the 6^{th}-degree partial sum for the general power series method.*

c. *Which of the two approaches just used is simpler?*

Some Answers to Some of the Exercises

WARNING! Most of the following answers were prepared hastily and late at night. They have not been properly proofread! Errors are likely!

6a. No singular points

6b. $z = \left[n + \frac{1}{2}\right]\pi$ with $n = 0, \pm 1, \pm 2, \ldots$

6c. $z = 0, \pm 1, \pm 2, \pm 3, \ldots$

6d. $z = in\pi$ with $n = 0, \pm 1, \pm 2, \ldots$

6e. $z = in\pi$ with $n = \pm 1, \pm 2, \ldots$

6f. $z = \pm 2i$

6g. $z = in2\pi$ with $n = 0, \pm 1, \pm 2, \ldots$

6h. $z = -2$

6i. no singular points

6j. $z = \pm\sqrt{n}$ with $n = 1, 2, 3, \ldots$

7a. $a_0\left[1 + x + x^2 + \frac{5}{6}x^3 + \frac{5}{8}x^4\right]$, $(-\infty, \infty)$

7b. $a_0\left[1 - x - \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{3}{8}x^4\right]$, $(-\infty, \infty)$

7c. $a_0\left[1 - x - \frac{1}{2}x^2 - \frac{1}{8}x^4\right]$, $(-\infty, \infty)$

7d. $a_0\left[1 - \frac{1}{2}(x-1)^2 + \frac{1}{6}(x-1)^3 + \frac{1}{24}(x-1)^4\right]$, $(0, 2)$

8a. $a_0\left[1 + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{12}x^4\right] + a_1\left[x + \frac{1}{6}x^3 + \frac{1}{12}x^4\right]$, $(-\infty, \infty)$

8b. $a_0\left[1 + \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{1}{6}x^4\right] + a_1\left[x - \frac{1}{3}x^3 + \frac{1}{12}x^4\right]$, $(-\infty, \infty)$

8c. $a_0\left[1 - \frac{1}{2}x^2 + \frac{1}{18}x^4\right] + a_1\left[x - \frac{1}{6}x^3\right]$, $(-\infty, \infty)$

8d. $a_0\left[1 - \frac{1}{6}(x-1)^3 + \frac{1}{24}(x-1)^4\right] + a_1\left[(x-1) - \frac{1}{12}(x-1)^4\right]$, $(0, 2)$

8e. $a_0\left[1 - \frac{1}{2}(x-1)^2 + \frac{1}{12}(x-1)^3 + \frac{1}{96}(x-1)^4\right] + a_1\left[(x-1) - \frac{1}{6}(x-1)^3 + \frac{1}{24}(x-1)^4\right]$, $(0, 2)$

8f. $a_0\left[1 - x^2 - \frac{5}{3}x^3 + \frac{11}{12}x^4\right] + a_1\left[x - \frac{1}{2}x^2 - \frac{1}{2}x^3 - \frac{9}{8}x^4\right]$, $(-\infty, \infty)$

9b i. $a_0\left[1 + x + x^2 + \frac{5}{6}x^3 + \frac{5}{8}x^4 + \frac{13}{30}x^5 + \frac{203}{720}x^6 + \frac{877}{5040}x^7 + \frac{23}{224}x^8 + \frac{1007}{17280}x^9 + \frac{4639}{145152}x^{10}\right]$

9b ii. $a_0\left[1 - x + \frac{1}{2}x^2 - \frac{1}{3}x^3 + \frac{5}{24}x^4 - \frac{1}{15}x^5 + \frac{13}{720}x^6 - \frac{11}{630}x^7 + \frac{361}{40320}x^8\right]$

9b iii. $a_0\left[1 - x + \frac{1}{2}x^2 - \frac{1}{3}x^3 + \frac{5}{24}x^4 - \frac{2}{15}x^5 + \frac{61}{720}x^6 - \frac{17}{315}x^7 + \frac{277}{8064}x^8\right]$

9b iv. $a_0\left[1 - 3(x-2) + \frac{23}{6}(x-2)^2 - \frac{407}{162}(x-2)^3 + \frac{1241}{1944}(x-2)^4 + \frac{21629}{87480}(x-2)^5\right]$

10b i. $a_0\left[1 + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{12}x^4 + \frac{1}{24}x^5 + \frac{13}{720}x^6 + \frac{1}{140}x^7 + \frac{109}{40320}x^8\right]$
$+ a_1\left[x + \frac{1}{6}x^3 + \frac{1}{12}x^4 + \frac{1}{30}x^5 + \frac{1}{72}x^6 + \frac{29}{5040}x^7 + \frac{1}{448}x^8\right]$

10b ii. $a_0\left[1 - \frac{1}{2}x^2 + \frac{1}{12}x^4 - \frac{1}{80}x^6 + \frac{11}{8064}x^8 - \frac{17}{129600}x^{10}\right]$
$+ a_1\left[x - \frac{1}{6}x^3 + \frac{1}{30}x^5 - \frac{19}{5040}x^7 + \frac{29}{72576}x^9\right]$

10b iii. $a_0\left[1 - \frac{1}{2}x^2 + \frac{1}{6}x^4 - \frac{31}{720}x^6\right] + a_1\left[x - \frac{1}{3}x^3 + \frac{1}{10}x^5 - \frac{59}{2520}x^7\right]$

10b iv. $a_0\left[1 - \frac{1}{2}(x-1)^2 + \frac{1}{12}(x-1)^3 - \frac{1}{96}(x-1)^4 + \frac{31}{960}(x-1)^5\right]$

$$+ a_1 \left[(x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{12}(x-2)^3 - \frac{3}{32}(x-1)^4 + \frac{71}{960}(x-4)^5 \right]$$

11a i. $a_k = \frac{1}{3} a_{k-1}$ for $k \geq 1$

11a ii. $a_0 \left[1 + \frac{1}{3}x + \frac{1}{3^2}x^2 + \frac{1}{3^3}x^3 + \frac{1}{3^4}x^4 + \frac{1}{3^5}x^5 + \frac{1}{3^6}x^6 \right]$

11b i. $a_k = \frac{1}{k} \sum_{j=0}^{k-1} \frac{1}{3^{k-j}} a_j$ for $k \geq 0$

11b ii. $a_0 \left[1 + \frac{1}{3}x + \frac{1}{3^2}x^2 + \frac{1}{3^3}x^3 + \frac{1}{3^4}x^4 + \frac{1}{3^5}x^5 + \frac{1}{3^6}x^6 \right]$