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The Big Theorem on the Frobenius Method, With Applications

At this point, you may have a number of questions, including:

1. What do we do when the basic method does not yield the necessary linearly independent pair of solutions?
2. Are there any shortcuts?

To properly answer these questions requires a good bit of analysis — some straightforward and some, perhaps, not so straightforward. We will do that in the next chapter. Here, instead, we will present a few theorems summarizing the results of that analysis, and we will see how those results can, in turn, be applied to solve and otherwise gain useful information about solutions to some notable differential equations.

By the way, in the following, it does not matter whether we are restricting ourselves to differential equations with rational coefficients, or are considering the more general case. The discussion holds for either.

34.1 The Big Theorems

The Theorems

The first theorem simply restates some results discussed earlier in section 33.5.

Theorem 34.1 (the indicial equation and corresponding exponents)
Let x_0 be a point on the real line. Then x_0 is a regular singular point for a given second-order, linear homogeneous differential equation if and only if that differential equation can be written as

$$(x - x_0)^2 \alpha(x) y'' + (x - x_0)\beta(x) y' + \gamma(x) y = 0$$

where α, β and γ are all analytic at x_0 with $\alpha(x_0) \neq 0$. Moreover:

1. *The indicial equation arising in the method of Frobenius to solve this differential equation is*

$$\alpha_0 r(r-1) + \beta_0 r + \gamma_0 = 0$$

where

$$\alpha_0 = \alpha(x_0) \quad , \quad \beta_0 = \beta(x_0) \quad \text{and} \quad \gamma_0 = \gamma(x_0) \quad .$$

2. *The indicial equation has exactly two solutions r_1 and r_2 (possibly identical). And, if $\alpha(x_0)$, $\beta(x_0)$ and $\gamma(x_0)$ are all real valued, then r_1 and r_2 are either both real valued or are complex conjugates of each other.*

The next theorem is "the big theorem" of the Frobenius method. It describes generic formulas for solutions about regular singular points, and gives the intervals over which these formulas are valid.

Theorem 34.2 (general solutions about regular singular points)
Assume x_0 is a regular singular point on the real line for some given second-order homogeneous linear differential equation with real coefficients. Let R be the corresponding Frobenius radius of convergence, and let r_1 and r_2 be the two solutions to the corresponding indicial equation, with $r_1 \geq r_2$ if they are real. Then, on the intervals $(x_0, x_0 + R)$ and $(x_0 - R, x_0)$, general solutions to the differential equation are given by

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

where c_1 and c_2 are arbitrary constants, and y_1 and y_2 are solutions that can be written as follows[1]*:*

1. *In general,*

$$y_1(x) = |x - x_0|^{r_1} \sum_{k=0}^{\infty} a_k (x - x_0)^k \qquad \text{with} \quad a_0 = 1 \quad . \tag{34.1}$$

2. *If $r_1 - r_2$ is* not *an integer, then*

$$y_2(x) = |x - x_0|^{r_2} \sum_{k=0}^{\infty} b_k (x - x_0)^k \qquad \text{with} \quad b_0 = 1 \quad . \tag{34.2}$$

3. *If $r_1 - r_2 = 0$ (i.e., $r_1 = r_2$), then*

$$y_2(x) = y_1(x) \ln|x - x_0| + |x - x_0|^{1+r_1} \sum_{k=0}^{\infty} b_k (x - x_0)^k \quad . \tag{34.3}$$

4. *If $r_1 - r_2 = K$ for some positive integer K, then*

$$y_2(x) = \mu y_1(x) \ln|x - x_0| + |x - x_0|^{r_2} \sum_{k=0}^{\infty} b_k (x - x_0)^k \quad . \tag{34.4}$$

where $b_0 = 1$, b_K is arbitrary and μ is some (nonarbitrary) constant (possibly zero). Moreover,

$$y_2(x) = y_{2,0}(x) + b_K y_1(x)$$

where $y_{2,0}(x)$ is given by formula (34.4) with $b_K = 0$.

[1] In this theorem, we are assigning convenient values to the coefficients, such as a_0 and b_0, that could, in fact, be considered as arbitrary nonzero constants. Any coefficient not explicitly mentioned is not arbitrary.

Alternate Formulas
Solutions Corresponding to Integral Exponents

Remember that in developing the basic method of Frobenius, the first solution we obtained was actually of the form

$$y_1(x) = (x-x_0)^{r_1}\sum_{k=0}^{\infty} a_k(x-x_0)^k$$

and not

$$y_1(x) = |x-x_0|^{r_1}\sum_{k=0}^{\infty} a_k(x-x_0)^k$$

As noted on page 33–24, either solution is valid on both (x_0, x_0+R) and on (x_0-R, x_0). It's just that the second formula yields a real-valued solution even when $x < x_0$ and r_1 is a real number other than an integer.

Still, if r_1 is an integer — be it 8, 0, -2 or any other integer — then replacing

$$(x-x_0)^{r_1} \qquad \text{with} \qquad |x-x_0|^{r_1}$$

is completely unnecessary, and is usually undesirable. This is especially true if $r_1 = n$ for some nonnegative integer n, because then

$$y_1(x) = (x-x_0)^n\sum_{k=0}^{\infty} a_k(x-x_0)^k = \sum_{k=0}^{\infty} a_k(x-x_0)^{k+n}$$

is actually a power series about x_0, and the proof of theorem 34.2 (given later) will show that this is a solution on the entire interval (x_0-R, x_0+R). It might also be noted that, in this case, y_1 is analytic at x_0, a fact that might be useful in some applications.

Similar comments hold if the other exponent, r_2, is an integer. So let us make official:

Corollary 34.3 (solutions corresponding to integral exponents)
Let r_1 and r_2 be as in theorem 34.2.

1. *If r_1 is an integer, then the solution given by formula (34.1) can be replaced by the solution given by*

$$y_1(x) = (x-x_0)^{r_1}\sum_{k=0}^{\infty} a_k(x-x_0)^k \qquad \textit{with} \quad a_0 = 1 \quad . \tag{34.5}$$

2. *If r_2 is an integer while r_1 is not, then the solution given by formula (34.2) can be replaced by the solution given by*

$$y_2(x) = (x-x_0)^{r_2}\sum_{k=0}^{\infty} b_k(x-x_0)^k \qquad \textit{with} \quad b_0 = 1 \quad . \tag{34.6}$$

3. *If $r_1 = r_2$ and are integers, then the solution given by formula (34.3) can be replaced by the solution given by*

$$y_2(x) = y_1(x)\ln|x-x_0| + (x-x_0)^{1+r_1}\sum_{k=0}^{\infty} b_k(x-x_0)^k \quad . \tag{34.7}$$

4. *If r_1 and r_2 and are two different integers, then the solution given by formula (34.4) can be replaced by the solution given by*

$$y_2(x) = \mu y_1(x) \ln|x - x_0| + (x - x_0)^{r_2} \sum_{k=0}^{\infty} b_k (x - x_0)^k \quad . \tag{34.8}$$

Other Alternate Solutions Formulas

It should also be noted that alternative versions of formulas (34.3) and (34.4) can be found by simply factoring out $y_1(x)$ from both terms, giving us

$$y_2(x) = y_1(x) \left[\ln|x - x_0| + (x - x_0)^1 \sum_{k=0}^{\infty} c_k (x - x_0)^k \right] \tag{34.3$'$}$$

when $r_2 = r_1$, and

$$y_2(x) = y_1(x) \left[\mu \ln|x - x_0| + (x - x_0)^{-K} \sum_{k=0}^{\infty} c_k (x - x_0)^k \right] \tag{34.4$'$}$$

when $r_1 - r_2 = K$ for some positive integer K. In both cases, $c_0 = b_0$.

Theorem 34.2 and the Method of Frobenius

Remember that, using the basic method of Frobenius, we can find every solution of the form

$$y(x) = |x - x_0|^r \sum_{k=0}^{\infty} a_k (x - x_0)^k \qquad \text{with} \quad a_0 \neq 0 \quad ,$$

provided, of course, such solutions exist. Fortunately for us, statement 1 in the above theorem assures us that such solutions will exist corresponding to r_1. This means our basic method of Frobenius will successfully lead to a valid first solution (at least when the coefficients of the differential equation are rational). Whether there is a second solution y_2 of this form depends:

1. If $r_1 - r_2$ is not an integer, then statement 2 of the theorem states that there is such a second solution. Consequently, step 8 of the method will successfully lead to the desired result.

2. If $r_1 - r_2$ is a positive integer, then there might be such a second solution, depending on whether or not $\mu = 0$ in formula (34.4). If $\mu = 0$, then formula (34.4) for y_2 reduces to the sort of modified power series we are seeking, and step 8 in the Frobenius method will give us this solution. What's more, as indicated in statement 4 of the above theorem, in carrying out step 8, we will also rederive the solution already obtained in steps 7 (unless we set $b_K = 0$)!

   On the other hand, if $\mu \neq 0$, then it follows from the above theorem that no such second solution exists. As a result, all the work carried out in step 8 of the basic Frobenius method will lead only to a disappointing end; namely, that the terms "blow up" (as discussed in the subsection *Problems Possibly Arising in Step 8* starting on page 33–29).[2]

3. Of course, if $r_2 = r_1$, then the basic method of Frobenius cannot yield a second solution different from the first. But our theorem does assure us that we can use formula (34.3) for y_2 (once we figure out the values of the b_k's).

So what can we do if the basic method of Frobenius does not lead to the second solution $y_2(x)$? At this point, we have two choices, both using the formula for $y_1(x)$ already found by the basic Frobenius method:

1. Use the reduction of order method.
2. Plug formula (34.3) or (34.4), as appropriate, into the differential equation, and solve for the b_k's (and, if appropriate, μ).

In practice — especially if all you have for $y_1(x)$ is the modified power series solution from the basic method of Frobenius — you will probably find that the second of the above choices to be the better choice. It does lead, at least, to a usable recursion formula for the b_k's . We will discuss this further in section 34.6.

However, as we'll discuss in the next few sections, you might not need to find that y_2 .

34.2 Local Behavior of Solutions: Issues

In many situations, we are interested in knowing something about the general behavior of a solution $y(x)$ to a given differential equation when x is at or near some point x_0 . Certainly, for example, we were interested in what $y(x)$ and $y'(x)$ became as $x \to x_0$ when considering initial value problems.

In other problems (which we will see later), x_0 may be an endpoint of an interval of interest, and we may be interested in a solution y only if $y(x)$ and its derivatives remain well behaved (e.g., finite) as $x \to x_0$. This can become a very significant issue when x_0 is a singular point for the differential equation in question.

In fact, there are two closely related issues that will be of concern:

1. What does $y(x)$ approach as $x \to x_0$? Must it be zero? Can it be some nonzero finite value? Or does $y(x)$ "blow up" as $x \to x_0$? (And what about the derivatives of $y(x)$ as $x \to x_0$?)
2. Can we treat y as being analytic at x_0 ?

Of course, if y is analytic at x_0 , then, for some $R > 0$, it can be represented by a power series

$$y(x) = \sum_{k=0}^{\infty} c_k (x - x_0)^k \qquad \text{for} \quad |x - x_0| < R \quad ,$$

[2] We will find a formula for μ in the next chapter. Unfortunately, it is not a simple formula, and is not of much help in preventing us from attempting step 8 of the basic Frobenius method when that step does not lead to y_2 .

and, as we already know,

$$\lim_{x\to x_0} y(x) = y(x_0) = c_0 \qquad \text{and} \qquad \lim_{x\to x_0} y'(x) = y'(x_0) = c_1 \quad .$$

Thus, when y is analytic at x_0, the limits in question are "well behaved", and the question of whether $y(x_0)$ must be zero or can be some nonzero value reduces to the question of whether c_0 is or is not zero.

In the next two sections, we will examine the possible behavior of solutions to a differential equation near a regular singular point for that equation. Our ultimate interest will be in determining when the solutions are "well behaved" and when they are not. Then, in section 34.5, we will apply our results to an important set of equations from mathematical physics, the Legendre equations.

34.3 Local Behavior of Solutions: Limits at Regular Singular Points

As just noted, a major concern in many applications is the behavior of a solution $y(x)$ as x approaches a regular singular point x_0. In particular, it may be important to know whether

$$\lim_{x\to x_0} y(x)$$

is zero, some finite nonzero value, infinite, or completely undefined.

We discussed this issue at the beginning of chapter 31 for the shifted Euler equation

$$(x-x_0)^2\alpha_0 y'' + (x-x_0)\beta_0 y' + \gamma_0 y = 0 \quad . \tag{34.9}$$

Let us try using what we know about those solutions after comparing them with the solutions given in our big theorem for

$$(x-x_0)^2\alpha(x)y'' + (x-x_0)\beta(x)y' + \gamma(x)y = 0 \tag{34.10}$$

assuming

$$\alpha_0 = \alpha(x_0) \quad , \quad \beta_0 = \beta(x_0) \quad \text{and} \quad \gamma_0 = \gamma(x_0) \quad .$$

Remember, these two differential equations have the same indicial equation

$$\alpha_0 r(r-1) + \beta_0 r + \gamma_0 = 0 \quad .$$

Naturally, we'll further assume α, β and γ are analytic at x_0 with $\alpha(x_0) \neq 0$ so we can use our theorems. Also, in the following, we will let r_1 and r_2 be the two solutions to the indicial equation, with $r_1 \geq r_2$ if they are real.

Preliminary Approximations

Observe that each of the solutions described in theorem 34.2 involves one or more power series

$$\sum_{k=0}^{\infty} c_k(x-x_0)^k = c_0 + c_1(x-x_0) + c_2(x-x_0)^2 + c_3(x-x_0)^3 + \cdots$$

where $c_0 \neq 0$. As we've noted in earlier chapters,

$$\lim_{x \to x_0} \sum_{k=0}^{\infty} c_k (x - x_0)^k = \lim_{x \to x_0} \left[c_0 + c_1(x - x_0) + c_2(x - x_0)^2 + \cdots \right] = c_0 \quad .$$

This means

$$\sum_{k=0}^{\infty} c_k (x - x_0)^k \approx c_0 \qquad \text{when} \quad x \approx x_0 \quad .$$

Solutions Corresponding to r_1

Now, recall that the solutions corresponding to r_1 for the shifted Euler equation are constant multiples of

$$y_{\text{Euler},1}(x) = |x - x_0|^{r_1} \quad ,$$

while the corresponding solutions to equation (34.10) are constant multiples of something of the form

$$y_1(x) = |x - x_0|^{r_1} \sum_{k=0}^{\infty} a_k (x - x_0)^k \qquad \text{with} \quad a_0 = 1 \quad .$$

Using the approximation just noted for the power series, we immediately have

$$y_1(x) \approx |x - x_0|^{r_1} a_0 = |x - x_0|^{r_1} = y_{\text{Euler},1}(x) \qquad \text{when} \quad x \approx x_0 \quad .$$

which is exactly what we suspected back the beginning of section 33.3. In particular, if r_1 is real,

$$\lim_{x \to x_0} |y_1(x)| = \lim_{x \to x_0} |x - x_0|^{r_1} = \begin{cases} 0 & \text{if} \quad r_1 > 0 \\ 1 & \text{if} \quad r_1 = 0 \\ +\infty & \text{if} \quad r_1 < 0 \end{cases} \quad .$$

(For the case where r_1 is not real, see exercise 34.3.)

Solutions Corresponding to r_2 when $r_2 \neq r_1$

In this case, all the corresponding solutions to the shifted Euler equation are given by constant multiples of

$$y_{\text{Euler},2} = |x - x_0|^{r_2} \quad .$$

If r_1 and r_2 do not differ by an integer, then the corresponding solutions to equation (34.10) are constant multiples of something of the form

$$y_2(x) = |x - x_0|^{r_2} \sum_{k=0}^{\infty} a_k (x - x_0)^k \qquad \text{with} \quad a_0 = 1 \quad ,$$

and the same arguments given above with $r = r_1$ apply and confirm that

$$y_2(x) \approx |x - x_0|^{r_2} a_0 = |x - x_0|^{r_2} = y_{\text{Euler},2}(x) \qquad \text{when} \quad x \approx x_0 \quad .$$

On the other hand, if $r_1 - r_2 = K$ for some positive integer K, then the corresponding solutions to (34.10) are constant multiples of something of the form

$$y_2(x) = \mu y_1(x) \ln|x - x_0| + |x - x_0|^{r_2} \sum_{k=0}^{\infty} b_k (x - x_0)^k \qquad \text{with} \quad b_1 = 1 \quad .$$

Using approximations already discussed, we have, when $x \approx x_0$,

$$\begin{aligned} y_2(x) &\approx \mu\, |x - x_0|^{r_1} \ln|x - x_0| + |x - x_0|^{r_2}\, b_0 \\ &= \mu\, |x - x_0|^{r_2 + K} \ln|x - x_0| + |x - x_0|^{r_2} \\ &= |x - x_0|^{r_2} \left[\mu\, |x - x_0|^{K} \ln|x - x_0| + 1\right] \quad . \end{aligned}$$

But K is a positive integer, and you can easily show, via L'Hôpital's rule, that

$$\lim_{x \to x_0} (x - x_0)^K \ln|x - x_0| = 0 \quad .$$

Thus, when $x \approx x_0$,

$$y_2(x) \approx |x - x_0|^{r_2} [\mu \cdot 0 + 1] = |x - x_0|^{r_2} \quad ,$$

confirming that, whenever $r_2 \neq r_1$

$$y_2(x) \approx y_{\text{Euler},2}(x) \qquad \text{when} \quad x \approx x_0 \quad .$$

In particular, if r_2 is real,

$$\lim_{x \to x_0} |y_2(x)| = \lim_{x \to x_0} |x - x_0|^{r_1} = \begin{cases} 0 & \text{if} \quad r_2 > 0 \\ 1 & \text{if} \quad r_2 = 0 \\ +\infty & \text{if} \quad r_2 < 0 \end{cases} \quad .$$

Solutions Corresponding to r_2 when $r_2 = r_1$

In this case, all the second solutions to the shifted Euler equation are constant multiples of

$$y_{\text{Euler},2} = |x - x_0|^{r_2} \ln|x - x_0| \quad ,$$

and the corresponding solutions to our original differential equation are constant multiples of

$$y_2(x) = y_1(x) \ln|x - x_0| + |x - x_0|^{1 + r_1} \sum_{k=0}^{\infty} b_k (x - x_0)^k \quad .$$

If $x \approx x_0$, then

$$\begin{aligned} y_2(x) &\approx |x - x_0|^{r_1} \ln|x - x_0| + |x - x_0|^{1 + r_1}\, b_0 \\ &= |x - x_0|^{r_1} \ln|x - x_0| \left[1 + \frac{|x - x_0|}{\ln|x - x_0|} b_0\right] \quad . \end{aligned}$$

But

$$\lim_{x \to x_0} \frac{|x - x_0|}{\ln|x - x_0|} = 0 \quad .$$

Consequently, when $x \approx x_0$,

$$y_2(x) \approx |x - x_0|^{r_1} \ln|x - x_0| \, [1 + 0 \cdot b_0] = |x - x_0|^{r_1} \ln|x - x_0| \quad ,$$

confirming that, again, we have

$$y_2(x) \approx y_{\text{Euler},2}(x) \qquad \text{when} \quad x \approx x_0 \quad .$$

Taking the limit (possibly using L'Hôpital's rule), you can then easily show that

$$\lim_{x \to x_0} |y_2(x)| = \lim_{x \to x_0} |a_0| \, |x - x_0|^{r_1} \, |\ln|x - x_0|| = \begin{cases} 0 & \text{if} \quad r_1 > 0 \\ +\infty & \text{if} \quad r_1 \leq 0 \end{cases} \quad .$$

!►*Example 34.1 (Bessel's equation of order* 1*):* *Suppose we are only interested in those solutions on* $(0, \infty)$ *to Bessel's equation of order* 1 *,*

$$x^2 y'' + x y' + \left(x^2 - 1\right) y = 0 \quad ,$$

that do not "blow up" as $x \to 0$ *.*

First of all, observe that this equation is already in the form

$$(x - x_0)^2 \alpha(x) y'' + (x - x_0) \beta(x) y' + \gamma(x) y = 0$$

with $x_0 = 0$ *,* $\alpha(x) = 1$ *,* $\beta(x) = 1$ *and* $\gamma(x) = x^2 - 1$ *. So* $x_0 = 0$ *is a regular singular point for this differential equation, and the indicial equation,*

$$\alpha_0 r(r - 1) + \beta_0 r + \gamma_0 = 0 \quad ,$$

is

$$1 \cdot r(r - 1) + 1 \cdot r - 1 = 0 \quad .$$

This simplifies to

$$r^2 - 1 = 0 \quad ,$$

and has solutions $r = \pm 1$ *. Thus, the exponents for our differential equation are*

$$r_1 = 1 \qquad \text{and} \qquad r_2 = -1 \quad .$$

By the analysis given above, we know that all solutions corresponding to r_1 *are constant multiples of one particular solution* y_1 *satisfying*

$$\lim_{x \to x_0} |y_1(x)| = \lim_{x \to x_0} |x - x_0|^1 = 0 \quad .$$

So none of the solutions corresponding to $r = 1$ *blow up as* $x \to 0$ *.*

On the other hand, the analysis given above for solutions corresponding to r_2 *when* $r_2 \neq r_1$ *tells us that all the (nontrivial) second solutions are (nonzero) constant multiples of one particular solution* y_2 *satisfying*

$$\lim_{x \to x_0} |y_2(x)| = \lim_{x \to x_0} |x - x_0|^{-1} = \infty \quad .$$

And these do "blow up" as $x \to 0$ *.*

Thus, since we are only interested in the solutions that do not blow up as $x \to 0$ *, we need only concern ourselves with those solutions corresponding to* $r_1 = 1$ *. Those corresponding to* $r_2 = -1$ *are not relevant, and we need not spend time and effort finding their formulas.*

Derivatives

To analyze the behavior of the derivative $y'(x)$ as $x \to x_0$, you simply differentiate the modified power series for $y(x)$, and then apply the ideas described above. I'll let you do it in exercise 34.4 at the end of the chapter.

34.4 Local Behavior: Analyticity and Singularities in Solutions

It is certainly easier to analyze how $y(x)$ or any of it's derivatives behave as $x \to x_0$ when $y(x)$ is given by a basic power series about x_0

$$y(x) = \sum_{k=0}^{\infty} c_k (x - x_0)^k \quad .$$

Then, in fact, y is infinitely differentiable at x_0, and we know

$$\lim_{x \to x_0} y(x) = y(x_0) = c_0$$

and

$$\lim_{x \to x_0} y(x) = y^{(k)}(x_0) = k!\, c_k \qquad \text{for} \quad k = 1, 2, 3, \ldots \quad .$$

On the other hand, if $y(x)$ is given by one of those modified power series described the big theorem, then, as we partly saw in the last section, computing the limits of $y(x)$ and its derivatives as $x \to x_0$ is much more of a challenge. Indeed, unless that series is of the form

$$(x - x_0)^r \sum_{k=0}^{\infty} \alpha_k (x - x_0)^k$$

with r being zero or some postive integer, then $\lim_{x \to x_0} y(x)$ may blow up or otherwise fail to exist. And even if that limit does exist, you can easily show that the corresponding limit of the n^{th} derivative, $y^{(n)}(x)$, will either blow up or fail to exist if $n > r$.

To simplify our discussion, let's slightly expand our "ordinary/singular point" terminology so that it applies to any function y. Basically, we want to refer to a point x_0 as an *ordinary point for a function* y if y is analytic at x_0, and we want to refer to x_0 as a *singular point for* y if y is not analytic at x_0.

There are, however, small technical issues that must be taken into account. Typically, our function y is defined on some interval (α, β), possibly by some power series or modified power series. So, initially at least, we must confine our definitions of ordinary and singular points for y to points in or at the ends of this interval. To be precise, for any such point x_0, we will say that x_0 is an *ordinary point* for y if and only if there is a power series $\sum_{k=0}^{\infty} a_k (x - x_0)^k$ with a nonzero radius of convergence R such that,

$$y(x) = \sum_{k=0}^{\infty} a_k (x - x_0)^k$$

for all x in (α, β) with $|x - x_0| < R$. If no such power series exists, then we'll say x_0 is a *singular point* for y.

By the way, instead of referring to x_0 as an ordinary point for y, we may also refer to x_0 as a *point of analyticity* for y. And we may say that y has a *singularity* at x_0 instead of saying x_0 is a singular point for y.

Our interest, of course, is in the functions that are solutions to differential equations, and, as we discovered in previous chapters, solutions to the sort of differential equations we are considering are analytic at the ordinary points of those equations. So we immediately have:

Lemma 34.4

Let y be a solution on an interval (α, β) to some second-order homogeneous differential equation, and let x_0 be either in (α, β) or be one of the endpoints. Then

$$x_0 \text{ is an ordinary point for the differential equation} \quad \Longrightarrow \quad x_0 \text{ is an ordinary point for } y \quad .$$

Equivalently,

$$x_0 \text{ is a singular point for } y \quad \Longrightarrow \quad x_0 \text{ is a singular point for the differential equation} \quad .$$

This lemma does not say that y must have a singularity at each singular point of the differential equation. After all, if x_0 is a regular singular point, and the first solution r_1 of the indicial equation is a nonnegative integer, say, $r_1 = 3$, then our big theorem assures us of a solution y_1 given by

$$y_1(x) = (x - x_0)^3 \sum_{k=0}^{\infty} a_k (x - x_0)^k \quad ,$$

which is a true power series since

$$(x - x_0)^3 \sum_{k=0}^{\infty} a_k (x - x_0)^k = \sum_{k=0}^{\infty} a_k (x - x_0)^{k+3} = \sum_{n=0}^{\infty} c_k (x - x_0)^n$$

where

$$c_n = \begin{cases} 0 & \text{if} \quad n < 3 \\ a_{n-3} & \text{if} \quad n \geq 3 \end{cases} \quad .$$

Still, there are cases where one can be sure that a solution y to a given second-order linear homogeneous differential equation has at least one singular point. In particular, consider the situation in which y is given by a power series

$$y(x) = \sum_{k=0}^{\infty} c_k (x - x_0)^k \qquad \text{for} \quad |x - x_0| < R$$

when R is the radius of convergence, and is finite. Replacing x with the complex variable z,

$$y(z) = \sum_{k=0}^{\infty} c_k (z - x_0)^k$$

yields a power series with radius of convergence R for a complex-variable function $y(z)$ analytic at least on the disk of radius R about z_0. Now, it can be shown that there must then be a point

z_s on the edge of this disk at which $y(z)$ is not "well behaved". Unsurprisingly, it can also be shown that this z_s is a singular point for the differential equation. And if all the singular points of this differential equation happen to lie on the real line, then this singular point z_s must be one of the two points on the real line satisfying $|z_s - x_0| = R$, namely,

$$z_s = x_0 - R \qquad \text{or} \qquad z_s = x_0 + R \quad .$$

That gives us the following theorem (which we will find useful in the following section and in some of the exercises).

Theorem 34.5
Let x_0 be a point on the real line, and assume

$$y(x) = \sum_{k=0}^{\infty} c_k (x - x_0)^k \qquad \text{for} \quad |x - x_0| < R$$

is a power power series solution to some homogeneous second-order differential equation. Suppose further, that both of the following hold:

1. *R is the radius of convergence for the above power series and is finite.*
2. *All the singular points for the differential equation are on the real axis.*

Then at least one of the two points $x_0 - R$ or $x_0 + R$ is a singular point for y.

Admittedly, our derivation of the above theorem was rather sketchy and involved claims that "can be shown". If that isn't good enough for you, then turn to section 34.7 for a more satisfying proof. There will be pictures.

34.5 A Case Study: The Legendre Equations

As an application of what we have just developed, let us analyze the behavior of the solutions to the set of Legendre equations. These equations often arise in physical problems in which some spherical symmetry can be expected, and in most applications (for reasons I cannot explain here[3] only those solutions that are "bounded" on $(-1, 1)$ are of interest. Our analysis here will allow us to readily find those solutions, and will save us from a lot of needless work dealing with those solutions that, ultimately, will not be of interest.

Let me remind you of what a *Legendre equation* is; it is any differential equation that can be written as

$$(1 - x^2)y'' - 2xy' + \lambda y = 0 \tag{34.11}$$

where λ, the equation's parameter, is some constant. You may recall these equations from exercise 31.9 on page 31–41, and we will begin our analysis here by recalling what we learned in that rather lengthy exercise.

[3] With luck, some of the reasons will be explained in a later chapter on "boundary-value problems" .

What We Already Know

In that exercise 31.9 on page 31–41, you (I hope) discovered the following:

1. The only singular points for each Legendre equation are $x = -1$ and $x = 1$.

2. For each λ, a general solution on $(-1, 1)$ of Legendre equation (34.11) is

$$y_\lambda(x) = a_0 y_{\lambda,E}(x) + a_1 y_{\lambda,O}(x)$$

where $y_{\lambda,E}(x)$ is a power series about 0 having just even-powered terms and whose first term is 1, and $y_{\lambda,O}(x)$ is a power series about 0 having just odd-powered terms and whose first term is x,

3. If $\lambda = m(m+1)$ for some nonnegative integer m, then the Legendre equation with parameter λ has polynomial solutions, all of which are constant multiples of the m^{th} degree polynomial $p_m(x)$ where

$$p_0(x) = y_{0,E}(x) = 1 \quad ,$$

$$p_1(x) = y_{2,O}(x) = x \quad ,$$

$$p_2(x) = y_{6,E}(x) = 1 - 3x^2 \quad ,$$

$$p_3(x) = y_{12,O}(x) = x - \frac{5}{3}x^3 \quad ,$$

$$p_4(x) = y_{20,E}(x) = 1 - 10x^2 + \frac{35}{3}x^4 \quad ,$$

$$p_5(x) = y_{30,O}(x) = x - \frac{14}{3}x^3 + \frac{21}{5}x^5$$

and, in general,

$$p_m(x) = \begin{cases} y_{\lambda,E}(x) & \text{if } m \text{ is even} \\ y_{\lambda,O}(x) & \text{if } m \text{ is odd} \end{cases} \quad .$$

4. The Legendre equation with parameter λ has no polynomial solution on $(-1, 1)$ if $\lambda \neq m(m+1)$ for every nonnegative integer m

5. If y is a nonpolynomial solution to a Legendre equation on $(-1, 1)$, then it is given by a power series about $x_0 = 0$ with a radius of convergence of exactly 1.

Let us now see what more we can determine about the solutions to the Legendre equations on $(-1, 1)$ using the material developed in the last few sections. For convenience, we will record the noteworthy results as we derive them in a series of lemmas which, ultimately, will be summarized in a major theorem on Legendre equations, theorem 34.11.

The Singular Points of the Solutions

First of all, we should note that, because polynomials are analytic everywhere, the polynomial solutions to Legendre's equation have no singular points.

Now, suppose y is a solution to a Legendre equation, but is not a polynomial. As noted above, $y(x)$ can be given by a power series about $x_0 = 0$ with radius of convergence 1. This, and the fact that the only singular points for Legendre's equation are the points $x = -1$ and $x = 1$ on the real line, means that lemma 34.5 applies and immediately gives us the following:

Lemma 34.6
Let y be a nonpolynomial solution to a Legendre equation on $(-1, 1)$. Then y must have a singularity at either $x = -1$ or at $x = 1$ (or both).

Solution Limits at $x = 1$

To determine the limits of the solutions at $x = 1$, we will first find the exponents of the Legendre equation at $x = 1$ by solving the appropriate indicial equation. To do this, it is convenient to first multiply the Legendre equation by -1 and factor the first coefficient, giving us

$$(x-1)(x+1)y'' + 2xy' - \lambda y = 0 \quad .$$

Multiplying this by $x - 1$ converts the equation into "standard form",

$$(x-1)^2\alpha(x)y'' + (x-1)\beta(x)y' + \gamma(x)y = 0$$

where

$$\alpha(x) = x + 1 \quad , \quad \beta(x) = 2x \quad \text{and} \quad \gamma(x) = -(x-1)\lambda \quad .$$

Thus, the corresponding indicial equation is

$$r(r-1)\alpha_0 + r\beta_0 + \gamma_0 = 0$$

with

$$\alpha_0 = \alpha(1) = 2 \quad , \quad \beta_0 = \beta(1) = 2 \quad \text{and} \quad \gamma_0 = \gamma(1) = 0 \quad .$$

That is, the exponents $r = r_1$ and $r = r_2$ must satisfy

$$r(r-1)2 + 2r = 0 \quad ,$$

which simplifies to $r^2 = 0$. So,

$$r_1 = r_2 = 0 \quad ,$$

and our big theorem on the Frobenius method, theorem 34.2 on page 34–2, tells us that any solution y to a Legendre equation on $(-1, 1)$ can be written as

$$y(x) = c_1^+ y_1^+(x) + c_2^+ y_2^+$$

where

$$y_1^+(x) = |x-1|^0 \sum_{k=0}^{\infty} a_k^+ (x-1)^k = \sum_{k=0}^{\infty} a_k^+ (x-1)^k \quad \text{with} \quad a_0^+ = 1 \quad ,$$

and

$$y_2^+(x) = y_1^+(x) \ln|x-1| + (x-1) \sum_{k=0}^{\infty} b_k^+ (x-1)^k \quad .$$

Now, let's make some simple observations using these formulas:

1. The point $x = 1$ is an ordinary point for the solution y_1^+. Moreover

$$\lim_{x \to 1} y_1^+(x) = a_0^+ = 1 \quad .$$

2. On the other hand, because of the logarithmic factor in y_2^+,

$$\lim_{x \to 1} y_2^+(x) = 1 \cdot \lim_{x \to 1} \ln|x - 1| + (1 - 1)b_0^+ = -\infty \quad .$$

Hence, $x = 1$ is not an ordinary point for y_2^+; it is a singular point for that solution.

3. More generally, if $y(x) = c_1^+ y_1^+(x) + c_2^+ y_2^+(x)$ is any nontrivial solution to Legendre's equation on $(-1, 1)$, then the following must hold:

(a) If $c_2^+ \neq 0$, then $x = 1$ is a singular point for y, and

$$\lim_{x \to 1} |y(x)| = \left| c_1^+ \underbrace{\lim_{x \to 1} y_1^+(x)}_{1} + c_2^+ \underbrace{\lim_{x \to 1} y_2^+(x)}_{-\infty} \right| = \infty \quad .$$

(b) If $c_2^+ = 0$, then $c_1^+ \neq 0$, and $x = 1$ is an ordinary point for y. Moreover,

$$\lim_{x \to 1} y(x) = c_1^+ \lim_{x \to 1} y_1^+(x) = c_1^+ \neq 0 \quad .$$

(Hence, $x = 1$ is a singular point for y if and only if $c_2^+ \neq 0$.)

All together, these observations give us:

Lemma 34.7
Let y be a nontrivial solution on $(-1, 1)$ to Legendre's equation. Then $x = 1$ is a singular point for y if and only if

$$\lim_{x \to 1^-} |y(x)| = \infty \quad .$$

Moreover, if $x = 1$ is not a singular point for y, then

$$\lim_{x \to 1^-} y(x)$$

exists and is a finite nonzero value.

Solution Limits at $x = -1$

A very similar analysis using $x = -1$ instead of $x = 1$ leads to:

Lemma 34.8
Let y be a nontrivial solution on $(-1, 1)$ to Legendre's equation. Then $x = -1$ is a singular point for y if and only if

$$\lim_{x \to -1^+} |y(x)| = \infty \quad .$$

Moreover, if $x = -1$ is not a singular point for y, then

$$\lim_{x \to -1^+} y(x)$$

exists and is a finite nonzero value.

?►*Exercise 34.1:* *Verify the above lemma by redoing the analysis done in the previous subsection using* $x = -1$ *in place of* $x = 1$ *.*

The Unboundedness of the Nonpolynomial Solutions

Recall that a function y is said to be *bounded* on an interval (a, b) if there is a finite number M which "bounds" the absolute value of $y(x)$ when x is in (a, b) ; that is,

$$|y(x)| \leq M \qquad \text{whenever} \quad a < x < b \quad .$$

Naturally, if a function is not bounded on the interval of interest, we say it is *unbounded*.

Now, if y happens to be one of those nonpolynomial solutions to a Legendre equation on $(-1, 1)$, then we know y has a singularity at either $x = -1$ or at $x = 1$ or at both (lemma 34.6). Lemmas 34.7 and 34.8 then tell us that

$$\lim_{x\to 1^-} |y(x)| = \infty \qquad \text{or} \qquad \lim_{x\to -1^+} |y(x)| = \infty \quad ,$$

clearly telling us that $y(x)$ is not bounded on $(-1, 1)$!

Lemma 34.9
Let y *be a nonzero solution to a Legendre equation on* $(-1, 1)$ *. If* y *is not a polynomial, then it is not bounded on* $(-1, 1)$ *.*

The Polynomial Solutions and Legendre Polynomials

Now assume y is a nonzero polynomial solution to a Legendre eqaution.

First of all, because each polynomial is a continuous function on the real line, we know from a classic theorem of calculus that the polynomial has maximum and minimum values over any given closed subinterval. Thus, in particular, our polynomial solution y has a maximum and a minimum value over $[-1, 1]$, and, hence, is bounded on $(-1, 1)$. So

Lemma 34.10
Each polynomial solution to a Legendre equation is bounded on $(-1, 1)$ *.*

But now remember that the Legendre equation with parameter λ has polynomial solutions if and only if $\lambda = m(m + 1)$ for some nonnegative integer m , and those solutions are all constant multiples of

$$p_m(x) = \begin{cases} y_{\lambda,E}(x) & \text{if} \quad m \text{ is even} \\ y_{\lambda,O}(x) & \text{if} \quad m \text{ is odd} \end{cases} \quad .$$

Since $x = 1$ is not a singular point for p_m , lemma 34.7 tells us that

$$p_m(1) = \lim_{x\to 1} p_m(1) \neq 0 \quad .$$

This allows us to define the m^{th} *Legendre polynomial* P_m by

$$P_m(x) = \frac{1}{p_m(1)} p_m(x) \qquad \text{for} \quad m = 0, 1, 2, 3, \ldots \quad .$$

Clearly, any constant multiple of p_m is also a constant multiple of P_m. So we can use the P_m's instead of the p_m's to describe all polynomial solutions to the Legendre equations.

In practice, it is more common to use the Legendre polynomials than the p_m's. In part, this is because

$$P_m(1) = \frac{1}{p_m(1)} p_m(1) = 1 \qquad \text{for} \quad m = 0, 1, 2, 3, \dots \quad .$$

With a little thought, you'll realize that this means that, for each nonnegative integer m, P_m is the polynomial solution to the Legendre equation with parameter $\lambda = m(m+1)$ that equals 1 when $x = 1$.

Summary

This is a good time to skim the lemmas in this section, along with the discussion of the Legendre polynomials, and verify for yourself that these results can be condensed into the following:

Theorem 34.11 (bounded solutions of the Legendre equations)
There are bounded, nontrivial solutions on $(-1, 1)$ to the Legendre equation

$$(1 - x^2)y'' - 2xy' + \lambda y = 0$$

if and only if $\lambda = m(m+1)$ for some nonnegative integer m. Moreover, y is such a solution if and only if y is a constant multiple of the m^{th} Legendre polynomial.

We may find a use for this theorem in a later chapter.

34.6 Finding Second Solutions Using Theorem 34.2

Let's return to actually solving differential equations.

When the basic method of Frobenius fails to deliver a second solution, we can turn to the appropriate formulas given in theorem 34.2 on page 34–2; namely, formula (34.3),

$$y_2(x) = y_1(x) \ln|x - x_0| + |x - x_0|^{1+r_1} \sum_{k=0}^{\infty} b_k (x - x_0)^k$$

or formula (34.4),

$$y_2(x) = \mu y_1(x) \ln|x - x_0| + |x - x_0|^{r_2} \sum_{k=0}^{\infty} b_k (x - x_0)^k \quad ,$$

depending, respectively, on whether the exponents r_1 and r_2 are equal or differ by a nonzero integer (the only cases for which the basic method might fail). The unknown constants in these formulas can then be found by fairly straightforward variations of the methods we've already developed: Plug formula (34.3) or (34.4) (as appropriate) into the differential equation, derive the recursion formula for the b_k's (and the value of μ if using formula (34.4)), and compute as many of the b_k's as desired.

Because the procedures are straightforward modifications of what we've already done many times in the last few chapters, we won't describe the steps in detail. Instead, we'll illustrate the basic ideas with an example, and then comment on those basic ideas. As you've come to expect in these last few chapters, the computations are simple but lengthy. But do note how the linearity of the equations is used to break the computations into more digestible pieces.

The Second Solution When $r_1 - r_2$ Is a Positive Integer

!►*Example 34.2:* *In example 33.9, starting on page 33–30, we attempted to find modified power series solutions about $x_0 = 0$ to*

$$2xy'' - 4y' - y = 0 \quad ,$$

which we rewrote as

$$2x^2y'' - 4xy' - xy = 0 \quad .$$

We found the exponents of this equation to be

$$r_1 = 3 \qquad \text{and} \qquad r_2 = 0 \quad ,$$

and obtained

$$y(x) = cx^3 \sum_{k=0}^{\infty} \frac{6}{2^k\, k!(k+3)!} x^k \tag{34.12}$$

as the solutions to the differential equation corresponding to r_1. Unfortunately, we found that there was not a similar solution corresponding to r_2.

To apply theorem 34.2, we first need the particular solution corresponding to r_1

$$y_1(x) = x^3 \sum_{k=0}^{\infty} a_k x^k \qquad \text{with} \quad a_0 = 1 \quad .$$

So we use formula (34.12) with c chosen so that

$$\underbrace{c\frac{6}{2^0\, 0!(0+3)!}}_{a_0} = 1 \quad .$$

Simple computations show that $c = 1$ and, so,

$$y_1(x) = x^3 \sum_{k=0}^{\infty} \frac{6}{2^k\, k!(k+3)!} x^k = \sum_{k=0}^{\infty} \frac{6}{2^k\, k!(k+3)!} x^{k+3} \quad .$$

According to theorem 34.2, the general solution to our differential equation is

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

where y_1 is as above, and (since $r_2 = 0$ and $K = r_1 - r_2 = 3$)

$$y_2(x) = \mu y_1(x) \ln|x - x_0| + \sum_{k=0}^{\infty} b_k x^k \quad .$$

where $b_0 = 1$, b_3 is arbitrary (we'll take it to be zero), and μ is some constant. To simplify matters, let us rewrite the last formula as

$$y_2(x) = \mu Y_1(x) + Y_2(x)$$

where

$$Y_1(x) = y_1(x)\ln|x| \qquad \text{and} \qquad Y_2(x) = \sum_{k=0}^{\infty} b_k x^k \quad .$$

Thus,

$$\begin{aligned} 0 &= 2x^2 {y_1}'' - 4x{y_1}' - xy_1 \\ &= 2x^2[\mu Y_1 + Y_2]'' - 4x[\mu Y_1 + Y_2]' - x[\mu Y_1 + Y_2] \quad . \end{aligned}$$

By the linearity of the derivatives, this can be rewritten as

$$0 = \mu\left\{2x^2{Y_1}'' - 4x{Y_1}' - xY_1\right\} + \left\{2x^2{Y_2}'' - 4x{Y_2}' - xY_2\right\} \quad . \tag{34.13}$$

Now, because y_1 *is a solution to our differential equation, you can easily verify that*

$$\begin{aligned} &2x^2{Y_1}'' - 4x{Y_1}' - xY_1 \\ &\qquad = 2x^2\frac{d^2}{dx^2}\left[y_1(x)\ln|x|\right] - 4x\frac{d}{dx}\left[y_1(x)\ln|x|\right] - x\left[y_1(x)\ln|x|\right] \\ &\qquad = \cdots \\ &\qquad = 4x{y_1}' - 6y_1 \quad . \end{aligned}$$

Replacing the y_1 *in the last line with its series formula then gives us*

$$2x^2{Y_1}'' - 4x{Y_1}' - xY_1 = 4x\frac{d}{dx}\sum_{k=0}^{\infty}\frac{6}{2^k\,k!(k+3)!}x^{k+3} - 6\sum_{k=0}^{\infty}\frac{6}{2^k\,k!(k+3)!}x^{k+3} \quad ,$$

which, after suitable computation and reindexing (you do it), becomes

$$2x^2{Y_1}'' - 4x{Y_1}' - xY_1 = \sum_{n=3}^{\infty}\frac{12(2n-3)}{2^{n-3}(n-3)!n!}x^n \quad . \tag{34.14}$$

Next, using the series formula for Y_2 *we have*

$$2x^2{Y_2}'' - 4x{Y_2}' - xY_2 = 2x^2\frac{d^2}{dx^2}\sum_{k=0}^{\infty}b_kx^k - 4x\frac{d}{dx}\sum_{k=0}^{\infty}b_kx^k - x\sum_{k=0}^{\infty}b_kx^k \quad ,$$

which, after suitable computations and changes of indices (again, you do it!), reduces to

$$2x^2{Y_2}'' - 4x{Y_2}' - xY_2 = \sum_{n=1}^{\infty}\left\{2n(n-3)b_n - b_{n-1}\right\}x^n \quad . \tag{34.15}$$

Combining equations (34.13), (34.14) and (34.15):

$$\begin{aligned} 0 &= \mu\left\{2x^2{Y_1}'' - 4x{Y_1}' - xY_1\right\} + \left\{2x^2{Y_2}'' - 4x{Y_2}' - xY_2\right\} \\ &= \mu\sum_{n=3}^{\infty}\frac{12(2n-3)}{2^{n-3}(n-3)!n!}x^n + \sum_{n=1}^{\infty}\left\{2n(n-3)b_n - b_{n-1}\right\}x^n \\ &= \mu\sum_{n=3}^{\infty}\frac{12(2n-3)}{2^{n-3}(n-3)!n!}x^n + \Bigg\{[-4b_1 - b_0]x^1 + [-4b_2 - b_1]x^2 \\ &\qquad\qquad + \sum_{n=3}^{\infty}\left\{2n(n-3)b_n - b_{n-1}\right\}x^n\Bigg\} \end{aligned}$$

$$= -[4b_1 + b_0]x^1 \;-\; [4b_2 + b_1]x^2 + \sum_{n=3}^{\infty}\left[2n(n-3)b_n - b_{n-1} + \mu\frac{12(2n-3)}{2^{n-3}(n-3)!n!}\right]x^n \quad .$$

Since each term in this last power series must be zero, we must have

$$-[4b_1 + b_0] \;=\; 0 \qquad , \qquad -[4b_2 + b_1] \;=\; 0$$

and

$$2n(n-3)b_n - b_{n-1} \;+\; \mu\frac{12(2n-3)}{2^{n-3}(n-3)!n!} \;=\; 0 \qquad \text{for} \quad n = 3,\ 4,\ 5,\ \dots \quad .$$

This (and the fact that we've set $b_0 = 1$ *) means that*

$$b_1 \;=\; -\frac{1}{4}b_0 \;=\; -\frac{1}{4}\cdot 1 \;=\; -\frac{1}{4} \qquad , \qquad b_2 \;=\; -\frac{1}{4}b_1 \;=\; -\frac{1}{4}\left(\frac{-1}{4}\right) \;=\; \frac{1}{16}$$

and

$$2n(n-3)b_n \;=\; b_{n-1} \;-\; \mu\frac{12(2n-3)}{2^{n-3}(n-3)!n!} \qquad \text{for} \quad n = 3,\ 4,\ 5,\ \dots \quad . \tag{34.16}$$

Because of the $n-3$ *factor in front of* b_n *, dividing the last equation by* $2n(n-3)$ *to get a recursion formula for the* b_n*'s would result in a recursion formula that "blow ups" for* $n = 3$ *. So we need to treat that case separately.*

With $n = 3$ *in equation (34.16), we get*

$$2\cdot 3(3-3)b_3 \;=\; b_2 \;-\; \mu\frac{12(2\cdot 3-3)}{2^{3-3}(3-3)!3!}$$

$$\hookrightarrow \qquad 0 \;=\; \frac{1}{16} \;-\; \mu 6$$

$$\hookrightarrow \qquad \mu \;=\; \frac{1}{96} \quad .$$

Notice that we obtained the value for μ *instead of* b_3*. As claimed in the theorem,* b_3 *is arbitrary. Because of this, and because we only need one second solution, let us set*

$$b_3 \;=\; 0 \quad .$$

Now we can divide equation (34.16) by $2n(n-3)$ *and use the value for* μ *just derived (with a little arithmetic) to obtain the recursion formula*

$$b_n \;=\; \frac{1}{2n(n-3)}\left[b_{n-1} \;-\; \frac{(2n-3)}{2^n(n-3)!n!}\right] \qquad \text{for} \quad n = 4,\ 5,\ 6,\ \dots \quad .$$

So,

$$b_4 \;=\; \frac{1}{2\cdot 4(4-3)}\left[b_3 \;-\; \frac{(2\cdot 4-3)}{2^4(4-3)!4!}\right] \;=\; \frac{1}{8}\left[0 \;-\; \frac{5}{384}\right] \;=\; -\frac{5}{3{,}072} \quad ,$$

$$b_5 \;=\; \frac{1}{2\cdot 5(5-3)}\left[b_4 \;+\; \frac{(2\cdot 5-3)}{2^5(5-3)!5!}\right] \;=\; \frac{1}{20}\left[\frac{-5}{3{,}072} \;+\; \frac{7}{7{,}680}\right] \;=\; -\frac{11}{307{,}200} \quad ,$$

$$\vdots$$

We won't attempt to find a general formula for the b_n's here!

Thus, a second particular solution to our differential equation is

$$y_2(x) = \mu y_1(x) \ln|x - x_0| + \sum_{k=0}^{\infty} b_k x^k$$

$$= \frac{1}{96} y_1(x) \ln|x| + \left\{1 - \frac{1}{4}x + \frac{1}{16}x^2 + 0x^3 - \frac{5}{3{,}072}x^4 - \frac{11}{307{,}200}x^5 + \cdots\right\}$$

where y_1 is our first particular solution,

$$y_1(x) = \sum_{k=0}^{\infty} \frac{6}{2^k\, k!(k+3)!} x^{k+3} \quad .$$

In general, when

$$r_1 - r_2 = K$$

for some positive integer K, the computations illustrated in the above example will yield a second particular solution. It will turn out that b_1, b_2, ... and b_{K-1} are all "easily computed" from b_0, just as in the example. You will also obtain a recursion relation for b_n in terms of lower-indexed b_k's and the coefficients from the series formula for y_1. This recursion formula (formula (34.16) in our example) will hold for $n \geq K$, but be degenerate when $n = K$ (just as in our example, where $K = 3$). From that degenerate case, the value of μ in formula (34.4) can be determined. The rest of the b_n's can then be computed using the recursion relation. Unfortunately, it is highly unlikely that you will be able to find a general formula for these b_n's in terms of just the index, n. So just compute as many as seem reasonable.

The Second Solution When $r_1 = r_2$

The basic ideas illustrated in the last example also apply when the exponents of the differential equation, r_1 and r_2, are equal. Of course, instead of using the formula used in the example for y_2, use formula (34.3),

$$y_2(x) = y_1(x) \ln|x - x_0| + |x - x_0|^{1+r_1} \sum_{k=0}^{\infty} b_k (x - x_0)^k \quad .$$

In this case, there is no "μ" to determine and none of the b_k's will be arbitrary. In some ways, that makes this a simpler case than considered in our example. I'll leave it to you to work out any details in the exercises.

34.7 Appendix on Proving Theorem 34.5

Our goal in this section is to describe how theorem 34.5 on page 34–12 can be rigorously verified. To aid us, we will first extend some of the material we developed on power series solutions and complex variables.

Complex Power Series Solutions

Throughout most of these chapters, we've been tacitly assuming that the derivatives in our differential equations are derivatives with respect to a real variable x,

$$y' = \frac{dy}{dx} \quad \text{and} \quad y'' = \frac{d^2y}{dx^2} \quad ,$$

as in elementary calculus. In fact, we can also have derivatives with respect to the complex variable z,

$$y' = \frac{dy}{dz} \quad \text{and} \quad y'' = \frac{d^2y}{dz^2} \quad .$$

The precise definitions for these "complex derivatives" are given in appendix 32.6 (see page 32–18), where it is pointed out that, computationally, differentiation with respect to z is completely analogous to the differentiation with respect to x learned in basic calculus. In particular, if

$$y(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k \quad \text{for} \quad |z - z_0| < R$$

for some point z_0 in the complex plane and some $R > 0$, then

$$\begin{aligned} y'(z) &= \frac{d}{dz} \sum_{k=0}^{\infty} a_k (z - z_0)^k \\ &= \sum_{k=0}^{\infty} \frac{d}{dz} \left[a_k (z - z_0)^k \right] = \sum_{k=1}^{\infty} k a_k (z - z_0)^{k-1} \quad \text{for} \quad |z - z_0| < R \quad . \end{aligned}$$

Consequently, all of our computations in chapters 31 and 32 can be carried out using the complex variable z instead of the real variable x, and using a point z_0 in the complex plane instead of a point x_0 on the real line. In particular, we have the following complex-variable analog of theorem 32.9 on page 32–10:

Theorem 34.12 (second-order series solutions)

Suppose z_0 is an ordinary point for a second-order homogeneous differential equation whose reduced form is

$$\frac{d^2y}{dz^2} + P\frac{dy}{dz} + Qy = 0 \quad .$$

Then P and Q have power series representations

$$P(z) = \sum_{k=0}^{\infty} p_k (z - z_0)^k \quad \text{for} \quad |z - z_0| < R$$

and

$$Q(z) = \sum_{k=0}^{\infty} q_k (z - z_0)^k \quad \text{for} \quad |z - z_0| < R$$

where R is the radius of analyticity about z_0 for this differential equation.

Moreover, a general solution to the differential equation is given by

$$y(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k \quad \text{for} \quad |z - z_0| < R$$

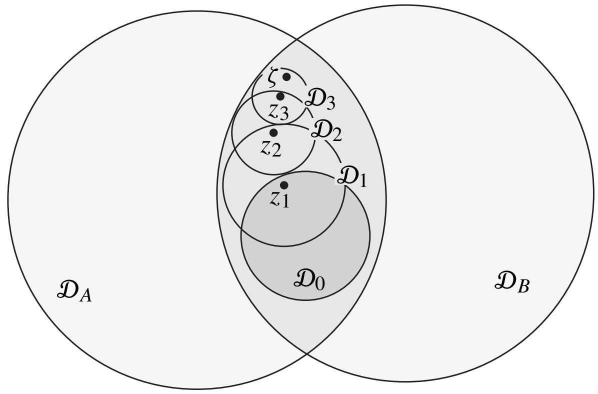

Figure 34.1: Disks in the complex plane for theorem 34.13 and its proof.

where a_0 and a_1 are arbitrary, and the other a_k's satisfy the recursion formula

$$a_k = -\frac{1}{k(k-1)} \sum_{j=0}^{k-2} \left[(j+1)a_{j+1}p_{k-2-j} + a_j q_{k-2-j} \right] \quad . \tag{34.17}$$

By the way, it should also be noted that, if

$$y(x) = \sum_{k=0}^{\infty} c_k (x - x_0)^k$$

is a solution to the real-variable differential equation for $|x - x_0| < R$, then

$$y(z) = \sum_{k=0}^{\infty} c_k (z - x_0)^k$$

is a solution to the corresponding complex-variable differential equation for $|z - x_0| < R$. This follows immediately from the last theorem and the relation between ${}^{dy}/_{dx}$ and ${}^{dy}/_{dz}$ (see the discussion of the complex derivative in section 32.6).

Analytic Continuation

Analytic continuation is any procedure that "continues" an analytic function defined on one region so that it becomes defined on a larger region. Perhaps it would be better to call it "analytic extension" because what we are really doing is extending the domain of our original function by creating an analytic function with a larger domain that equals the original function over the original domain.

We'll use two slightly different methods for doing analytic continuation. Both are based on Taylor series, and both can justified by the following theorem.

Theorem 34.13
Let $\mathcal{D}_A$ and $\mathcal{D}_B$ be two open disks in the complex plane that intersect each other, and let f_A be a function analytic on $\mathcal{D}_A$ and f_B a function analytic on $\mathcal{D}_B$. Assume further that there is an open disk $\mathcal{D}_0$ contained in both $\mathcal{D}_A$ and $\mathcal{D}_B$ (see figure 34.1), and that

$$f_A(z) = f_B(z) \qquad \text{for every} \quad z \in \mathcal{D}_0 \quad .$$

Then

$$f_A(z) = f_B(z) \qquad \text{for every} \quad z \in \mathcal{D}_A \cap \mathcal{D}_B \quad .$$

Think of f_A as being the original function defined on $\mathcal{D}_A$, and f_B as some other analytic function that we constructed on $\mathcal{D}_B$ to match f_A on $\mathcal{D}_0$. This theorem tells us that we can define a "new" function f on $\mathcal{D}_A \cup \mathcal{D}_B$ by

$$f(z) = \begin{cases} f_A(z) & \text{if} \quad z \text{ is in } \mathcal{D}_A \\ f_B(z) & \text{if} \quad z \text{ is in } \mathcal{D}_B \end{cases} \quad .$$

On the intersection, f is given both by f_A and f_B, but that is okay because the theorem assures us that f_A and f_B are the same on that intersection. And since f_A and f_B are, respectively, analytic at every point in $\mathcal{D}_A$ and $\mathcal{D}_B$, it follows that f is analytic on the union of $\mathcal{D}_A$ and $\mathcal{D}_B$, and satisfies

$$f(z) = f_A(z) \qquad \text{for each } z \text{ in } \mathcal{D}_A \quad .$$

That is, f is an "analytic extension" of f_A from the domain $\mathcal{D}_A$ to the domain $\mathcal{D}_A \cup \mathcal{D}_B$.

The proof of the above theorem is not difficult, and is somewhat instructive.

PROOF (theorem 34.13): We need to show that $f_A(\zeta) = f_B(\zeta)$ for every ζ in $\mathcal{D}_A \cap \mathcal{D}_B$. So let ζ be any point in $\mathcal{D}_A \cap \mathcal{D}_B$.

If $\zeta \in \mathcal{D}_0$ then, by our assumptions, we automatically have $f_A(\zeta) = f_B(\zeta)$.

On the other hand, if ζ in not in $\mathcal{D}_0$, then, as illustrated in figure 34.1, we can clearly find a finite sequence of open disks $\mathcal{D}_1$, $\mathcal{D}_2$, ... and $\mathcal{D}_M$ with respective centers z_1, z_2, ... and z_M such that

1. each z_k is also in $\mathcal{D}_{k-1}$,
2. each $\mathcal{D}_k$ is in $\mathcal{D}_A \cap \mathcal{D}_B$, and
3. the last disk, $\mathcal{D}_M$, contains ζ.

Now, because f_A and f_B are the same on $\mathcal{D}_0$, so are all their derivatives. Consequently, the Taylor series for f_A and f_B about the point z_1 in $\mathcal{D}_0$ will be the same. And since $\mathcal{D}_1$ is a disk centered at z_1 and contained in both $\mathcal{D}_A$ and $\mathcal{D}_B$, we have

$$\begin{aligned} f_A(z) &= \sum_{k=0}^{\infty} \frac{{f_A}^{(k)}(z_1)}{k!}(z - z_1)^k \\ &= \sum_{k=0}^{\infty} \frac{{f_B}^{(k)}(z_1)}{k!}(z - z_1)^k = f_B(z) \qquad \text{for every } z \text{ in } \mathcal{D}_1 \quad . \end{aligned}$$

Repeating these arguments using the Taylor series for f_A and f_B at the points z_2, z_3, and so on, we eventually get

$$f_A(z) = f_B(z) \qquad \text{for every } z \text{ in } \mathcal{D}_M \quad .$$

In particular then,

$$f_A(\zeta) = f_B(\zeta) \quad ,$$

just as we wished to show. ∎

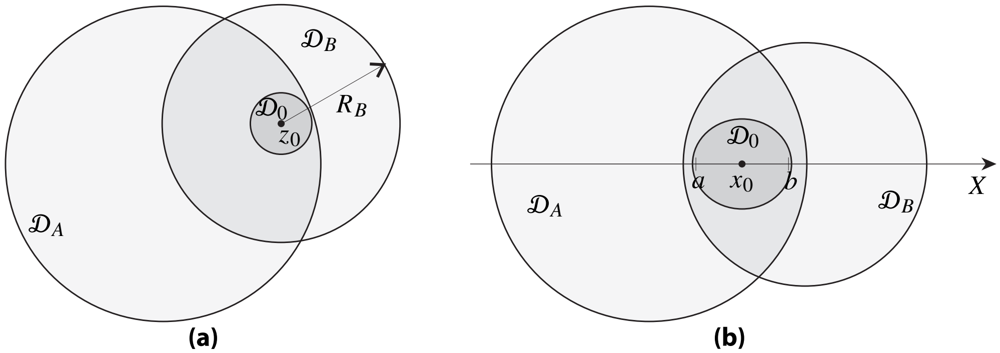

Figure 34.2: Disks in the complex plane for **(a)** corollary 34.14, in which a function on $\mathcal{D}_A$ is expanded to a function on $\mathcal{D}_B$ by a Taylor series at z_0, and **(b)** corollary 34.15, in which two functions are equal on the interval (a,b).

One standard way to do analytic continuation is to use the Taylor series of the original function f_A at a point z_0 to define the "second" function f_B on a disk $\mathcal{D}_B$ centered at z_0 with radius R_B. With luck, we can verify that R_B can be chosen so that $\mathcal{D}_B$ extends beyond the original disk, as in figure 34.2a. Now, since f_B is given by the Taylor series of f_A at z_0, theorem 32.14 on page 32–20 assures us that there is a disk $\mathcal{D}_0$ of positive radius about z_0 in $\mathcal{D}_A \cap \mathcal{D}_B$ on which f_A and f_B are equal (again, see figure 34.2a). A direct application of the theorem we've just proven then gives:

Corollary 34.14

Let f_A be a function analytic on some open disk $\mathcal{D}_A$ of the complex plane, and let z_0 be any point in $\mathcal{D}_A$. Set

$$f_B(z) = \sum_k^\infty c_k(z-z_0)^k \qquad \text{for} \quad |z-z_0| < R_B$$

where $\sum_k^\infty c_k(z-z_0)^k$ is the Taylor series for f_A about z_0 and R_B is the radius of convergence for this series. Then

$$f_A(z) = f_B(z) \qquad \text{for every} \quad z \in \mathcal{D}_A \cap \mathcal{D}_B \quad .$$

where $\mathcal{D}_B$ is the disk of radius R_B about z_0.

As the next corollary demonstrates, it may be possible to apply theorem 34.13 when the two functions are only known to be equal over an interval. This will be useful for us.

Corollary 34.15

Let $\mathcal{D}_A$ and $\mathcal{D}_B$ be two open disks in the complex plane that intersect each other, and let f_A be a function analytic on $\mathcal{D}_A$ and f_B a function analytic on $\mathcal{D}_B$. Assume further that there is an open interval (a,b) in the real axis contained in both $\mathcal{D}_A$ and $\mathcal{D}_B$, and that

$$f_A(x) = f_B(x) \qquad \text{for every} \quad x \in (a,b) \quad .$$

Then

$$f_A(z) = f_B(z) \qquad \text{for every} \quad z \in \mathcal{D}_A \cap \mathcal{D}_B \quad .$$

PROOF: Simply construct the Taylor series for f_A and f_B at some point x_0 in (a, b) (see figure 34.2b). Since these two functions are equal on this interval, so are their Taylor series at x_0. Hence, $f_A = f_B$ on some disk $\mathcal{D}_0$ of positive radius about x_0. The claim in the corollary then follows immediately from theorem 34.13. ∎

Proving Theorem 34.5

A Closely Related Lemma

Recall that theorem 34.5 makes a claim about the singular points of a power series solution based on the radius of convergence of that series. It turns out to be slightly more convenient to first prove the following lemma making a corresponding claim about the radius of convergence for a power series solution based on ordinary points of the solution.

Lemma 34.16

Let R be a finite positive value, and assume

$$y(x) = \sum_{k=0}^{\infty} c_k x^k \qquad \text{for} \quad -R < x < R \tag{34.18}$$

is a power power series solution to some homogeneous second-order differential equation. Suppose further, that all the singular points for this differential equation are on the real axis, but that $x = -R$ and $x = R$ are ordinary points for y. Then R is strictly less than the radius of convergence for the above series.

PROOF: By definition, $x = -R$ and $x = R$ being ordinary points for a function y defined on $(-R, R)$ means that there is a $\rho > 0$ and power series about $x = -R$ and $x = R$,

$$\sum_{k=0}^{\infty} c_k^{+}(x - R)^k \qquad \text{and} \qquad \sum_{k=0}^{\infty} c_k^{-}(x + R)^k \quad ,$$

such that

$$y(x) = \sum_{k=0}^{\infty} c_k^{+}(x - R)^k \qquad \text{for} \quad R - \rho < x < R$$

and

$$y(x) = \sum_{k=0}^{\infty} c_k^{-}(x + R)^k \qquad \text{for} \quad -R < x < -R + \rho \quad .$$

Now let $\mathcal{D}_-$ and $\mathcal{D}_+$ be the disks about $x = -R$ and $x = R$, respectively, each with radius ρ, and let

$$y^{+}(z) = \sum_{k=0}^{\infty} c_k^{+}(z - R)^k \qquad \text{for} \quad z \in \mathcal{D}_+$$

and

$$y^{-}(z) = \sum_{k=0}^{\infty} c_k^{-}(z + R)^k \qquad \text{for} \quad z \in \mathcal{D}_-$$

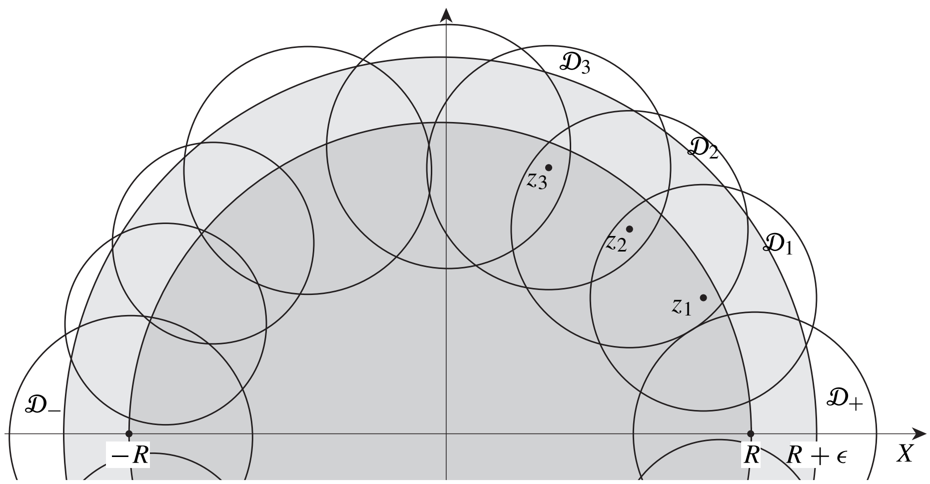

Figure 34.3: Disks for the proof of lemma 34.16. The darker disk is the disk on which y is originally defined.

(see figure 34.3).

Now, consider choosing a finite sequence of disks $\mathcal{D}_1$, $\mathcal{D}_2$, ..., and $\mathcal{D}_N$ with respective centers z_1 , z_2 , ..., and z_N such that that all the following holds:

1. Each z_n is inside the disk of radius R about the origin.

2. The radius of each $\mathcal{D}_n$ is no greater than the distance of z_n to the real axis.

3. The union of all the disks $\mathcal{D}_1$, $\mathcal{D}_2$, ..., and $\mathcal{D}_N$, along with the disks $\mathcal{D}_-$ and $\mathcal{D}_+$, contains not just the boundary of our original disk $\mathcal{D}$ but the "ring" of all z satisfying

$$R \leq |z| \leq R + \epsilon$$

for some positive value ϵ .

With a little thought, you will realize that these points and disks can be so chosen (as illustrated in figure 34.3).

Next, for each n , let y_n be the function given by the Taylor series of y at z_n ,

$$y_n(z) = \sum_{k=0}^{\infty} c_{n,k}(z - z_n)^k \qquad \text{with} \quad c_{n,k} = \frac{1}{k!} y^{(k)}(z_n) \quad .$$

Using the facts that y is a solution to the given differential equation on the disk with $|z| < R$, that the differential equation only has singular points on the real axis, and that $\mathcal{D}_n$ does not touch the real axis, we can apply theorem 34.12 in a straightforward manner to show that the above defines y_n as an analytic function on the entire disk $\mathcal{D}_n$. Repeated use of corollaries 34.14 and 34.15 then shows that any two functions from the set

$$\{y,\ y_+,\ y_-,\ y_1,\ y_2,\ \ldots,\ y_N\}$$

equal each other wherever both are defined. This allows us to define a "new" analytic function Y on the union of all of our disks via

$$Y(z) = \begin{cases} y(z) & \text{if} \quad |z| < R \\ y^{+}(z) & \text{if} \quad z \in \mathcal{D}_{+} \\ y^{-}(z) & \text{if} \quad z \in \mathcal{D}_{-} \\ y_n(z) & \text{if} \quad z \in \mathcal{D}_n \qquad \text{for} \quad n = 1, 2, \ldots, N \end{cases} .$$

Now, because the union of all of the disks contains the disk of radius $R + \epsilon$ about 0, theorem 32.14 on page 32–20 on the radius of convergence for Taylor series assures us that the Taylor series for Y about $z_0 = 0$ must have a radius of convergence of at least $R + \epsilon$. But, $y(z) = Y(z)$ when $|z| < R$. So y and Y have the same Taylor series at $z_0 = 0$. Thus, the radius of convergence for power series in equation (34.18), which is the Taylor series for y about $z_0 = 0$, must also be at least $R + \epsilon$, a value certainly greater than R. ∎

Clearly, a simple translation by x_0 will convert the last proof to a proof of

Lemma 34.17
Let x_0 and R be a finite real values with $R > 0$, and assume

$$y(x) = \sum_{k=0}^{\infty} c_k (x - x_0)^k \qquad \textit{for} \quad x_0 - R < x < x_0 + R$$

is a power power series solution to some homogeneous second-order differential equation. Suppose further, that all the singular points for this differential equation are on the real axis, but that $x = x_0 - R$ and $x = x_0 + R$ are ordinary points for y. Then R is strictly less than the radius of convergence for the above series.

Finally, a Short Proof of Theorem 34.5

Now suppose x_0 is a point on the real line, and

$$y(x) = \sum_{k=0}^{\infty} c_k (x - x_0)^k \qquad \text{for} \quad |x - x_0| < R$$

is a power power series solution to some homogeneous second-order differential equation. Suppose further, that both of the following hold:

1. R is the radius of convergence for the above power series and is finite.
2. All the singular points for the differential equation are on the real axis.

Our last lemma tells us that if $x_0 - R$ and $x_0 + R$ are both ordinary points for y, then R is not the radius of convergence for the above series for $y(x)$. But this contradicts the basic assumption that R is the radius of convergence for that series. Consequently, it is not possible for $x_0 - R$ and $x_0 + R$ to both be ordinary points for y. At least one must be a singular point, just as theorem 34.5 claims. ∎

Additional Exercises

34.2. *For each differential equation and singular point x_0 given below, let r_1 and r_2 be the corresponding exponents (with $r_1 \geq r_2$ if they are real), and let y_1 and y_2 be the two modified power series solutions about the given x_0 described in the "big theorem on the Frobenius method", theorem 34.2 on page 34–2, and do the following:*

i *If not already in the form*

$$(x - x_0)^2\alpha(x)y'' + (x - x_0)\beta(x)y' + \gamma(x)y = 0$$

where α, β and γ are all analytic at x_0 with $\alpha(x_0) \neq 0$, then rewrite the differential equation in this form.

ii *Determine the corresponding indicial equation, and find r_1 and r_2.*

iii *Write out the corresponding shifted Euler equation*

$$(x - x_0)^2\alpha(x_0)y'' + (x - x_0)\beta(x_0)y' + \gamma(x_0)y = 0 \quad .$$

and find the solutions $y_{\text{Euler},1}$ and $y_{\text{Euler},2}$ which approximate, respectively, $y_1(x)$ and $y_2(x)$ when $x \approx x_0$.

iv *Determine the limits $\lim_{x\to x_0} |y_1(x)|$ and $\lim_{x\to x_0} |y_2(x)|$.*

Do not attempt to find the modified power series formulas for y_1 and y_2.

a. $x^2y'' - 2xy' + \left(2 - x^2\right)y = 0 \quad , \quad x_0 = 0$

b. $x^2y'' - 2x^2y' + \left(x^2 - 2\right)y = 0 \quad , \quad x_0 = 0$

c. $y'' + \dfrac{1}{x}y' + y = 0 \quad , \quad x_0 = 0$ *(Bessel's equation of order 0)*

d. $x^2\left(2 - x^2\right)y'' + \left(5x + 4x^2\right)y' + (1 + x^2)y = 0 \quad , \quad x_0 = 0$

e. $x^2y'' - \left(5x + 2x^2\right)y' + 9y = 0 \quad , \quad x_0 = 0$

f. $x^2(1 + 2x)y'' + xy' + (4x^3 - 4)y = 0 \quad , \quad x_0 = 0$

g. $4x^2y'' + 8xy' + (1 - 4x)y = 0 \quad , \quad x_0 = 0$

h. $x^2y'' + xy' - (1 + 2x)y = 0 \quad , \quad x_0 = 0$

i. $xy'' + 4y' + \dfrac{12}{(x+2)^2}y = 0 \quad , \quad x_0 = 0$

j. $xy'' + 4y' + \dfrac{12}{(x+2)^2}y = 0 \quad , \quad x_0 = -2$

k. $(x - 3)y'' + (x - 3)y' + y = 0 \quad , \quad x_0 = 3$

l. $(1-x^2)y'' - xy' + 3y = 0 \quad , \quad x_0 = 1$ *(Chebyshev equation with parameter 3)*

34.3. *Suppose* x_0 *is a regular singular point for some second-order homogeneous linear differential equation, and that the corresponding exponents are complex*

$$r_+ = \lambda + i\omega \qquad \text{and} \qquad r_- = \lambda - i\omega$$

(with $\omega \neq 0$ *). Let* y *be any solution to this differential equation on an interval having* x_0 *as an endpoint. Show that*

$$\lim_{x \to x_0} y(x)$$

is zero if $\lambda > 0$ *, and does not exist if* $\lambda \leq 0$ *.*

34.4. *Assume* x_0 *is a regular singular point on the real line for*

$$(x - x_0)^2 \alpha(x) y'' + (x - x_0)\beta(x) y' + \gamma(x) y = 0$$

where, as usual, α *,* β *and* γ *are all analytic at* x_0 *with* $\alpha(x_0) \neq 0$ *. Assume the solutions* r_1 *and* r_2 *to the corresponding indicial equation are real, with* $r_1 \geq r_2$ *. Let* $y_1(x)$ *and* $y_2(x)$ *be the corresponding solutions to the differential equation as described in the big theorem on the Frobenius method, theorem 34.2.*

a. *Compute the derivatives of* y_1 *and* y_2 *and show that, for* $i = 1$ *and* $i = 2$ *,*

$$\lim_{x \to x_0} \left| y_i{}'(x) \right| = \begin{cases} 0 & \text{if } 1 < r_i \\ \infty & \text{if } 0 < r_i < 1 \\ \infty & \text{if } r_i < 0 \end{cases} .$$

Be sure to consider all cases.

b. *Compute* $\lim_{x \to x_0} \left| y_2{}'(x) \right|$ *when* $r_1 = 1$ *and when* $r_1 = 0$ *.*

c. *What can be said about* $\lim_{x \to x_0} \left| y_2{}'(x) \right|$ *when* $r_1 = 1$ *and when* $r_1 = 0$ *?*

34.5. *Recall that the Chebyshev equation with parameter* λ *is*

$$(1 - x^2) y'' - x y' + \lambda y = 0 \quad , \tag{34.19}$$

where λ *can be any constant. In exercise 31.10 on page 31–43 you discovered that:*

1. *The only singular points for each Chebyshev equation are* $x = 1$ *and* $x = -1$ *.*

2. *For each* λ *, the general solution on* $(-1, 1)$ *to equation (34.19) is given by*

$$y_\lambda(x) = a_0 y_{\lambda,E}(x) + a_1 y_{\lambda,O}(x)$$

where $y_{\lambda,E}$ *and* $y_{\lambda,O}$ *are, respectively, even- and odd-termed series*

$$y_{\lambda,E}(x) = \sum_{\substack{k=0 \\ k \text{ is even}}}^{\infty} c_k x^k \qquad \text{and} \qquad y_{\lambda,O}(x) = \sum_{\substack{k=0 \\ k \text{ is odd}}}^{\infty} c_k x^k$$

with $c_0 = 1$ *,* $c_1 = 1$ *and the other* c_k*'s determined from* c_0 *or* c_1 *via the recursion formula.*

3. *Equation (34.19) has nontrivial polynomial solutions if and only if* $\lambda = m^2$ *for some nonnegative integer* m *. Moreover, for each such* m *, all the polynomial solutions are constant multiples of an* m^{th} *degree polynomial* p_m *given by*

$$p_m(x) = \begin{cases} y_{\lambda,E}(x) & \text{if } m \text{ is even} \\ y_{\lambda,O}(x) & \text{if } m \text{ is odd} \end{cases} .$$

In particular,

$$p_0(x) = 1 \quad , \quad p_1(x) = x \quad , \quad p_2(x) = 1 - 2x^2 \quad ,$$

$$p_3(x) = x - \frac{4}{3}x^3 \quad , \quad p_4(x) = 1 - 8x^2 + 8x^4$$

and

$$p_5(x) = x - 4x^3 + \frac{16}{5}x^5$$

4. *Each nonpolynomial solution to a Chebyshev equation on* $(-1, 1)$ *is given by a power series about* $x_0 = 0$ *whose radius of convergence is exactly* 1 *.*

In the following, you will continue the analysis of the solutions to the Chebyshev equations in a manner analogous to our continuation of the analysis of the solutions to the Legendre equations in section 34.5.

a. *Verify that* $x = 1$ *and* $x = -1$ *are regular singular points for each Chebyshev equation.*

b. *Find the exponents* r_1 *and* r_2 *at* $x = 1$ *and* $x = -1$ *of each Chebyshev equation.*

c. *Let* y *be a nonpolynomial solution to a Chebyshev equation on* $(-1, 1)$ *, and show that either*

$$\lim_{x\to 1^-} \left|y'(x)\right| = \infty \qquad \text{or} \qquad \lim_{x\to -1^+} \left|y'(x)\right| = \infty$$

(or both limits are infinite).

d. *Verify that* $p_m(1) \neq 0$ *for each nonnegative integer* m *.*

e. *For each nonnegative integer* m *, the* m^{th} *Chebyshev polynomial* $T_m(x)$ *is the polynomial solution to the Chebyshev equation with parameter* $\lambda = m^2$ *satisfying* $T_m(1) = 1$ *. Find* $T_m(x)$ *for* $m = 0, 1, 2, \ldots, 5$ *.*

f. *Finish verifying that the Chebyshev equation*

$$(1 - x^2)y'' - 2xy' + \lambda y = 0$$

has nontrivial solutions with bounded first derivatives on $(-1, 1)$ *if and only if* $\lambda = m^2$ *for some nonnegative integer. Moreover,* y *is such a solution if and only if* y *is a constant multiple of the* m^{th} *Chebyshev polynomial.*

34.6. *The following differential equations all have* $x_0 = 0$ *as a regular singular point. For each, the corresponding exponents* r_1 *and* r_2 *are given, along with the solution* $y_1(x)$ *on* $x > 0$ *corresponding to* r_1 *, as described in theorem 34.2 on page 34–2. This* *solution can be found by the basic method of Frobenius. The second solution,* y_2 *, cannot be found be the basic method but, as stated in theorem 34.2 it is of the form*

$$y_2(x) \;=\; y_1(x)\ln|x| \;+\; |x|^{1+r_1}\sum_{k=0}^{\infty} b_k x^k$$

or

$$y_2(x) \;=\; \mu y_1(x)\ln|x| \;+\; |x|^{r_2}\sum_{k=0}^{\infty} b_k x^k \quad ,$$

depending, respectively, on whether the exponents r_1 *and* r_2 *are equal or differ by a nonzero integer. Do recall that, in the second formula* $b_0 = 1$ *, and one of the other* b_k*'s is arbitrary.*

"Find $y_2(x)$ *for* $x > 0$*" for each of the following. More precisely, determine which of the above two formulas hold, and find at least the values of* b_0 *,* b_1 *,* b_2 *,* b_3 *and* b_4 *, along with the value for* μ *if appropriate. You may set any arbitrary constant equal to* 0 *, and assume* $x > 0$ *.*

a. $4x^2y'' \;+\; (1-4x)y \;=\; 0 \quad : \quad r_1 = r_2 = \frac{1}{2} \quad , \quad y_1(x) \;=\; \sqrt{x}\sum_{k=0}^{\infty}\frac{1}{(k!)^2}x^k$

b. $y'' \;+\; \frac{1}{x}y' \;+\; y \;=\; 0$ *(Bessel's equation of order 0)* $\quad : \quad r_1 = r_2 = 0 \quad ,$

$$y_1(x) \;=\; \sum_{m=0}^{\infty}\frac{(-1)^m}{(2^m m!)^2}x^{2m}$$

c. $x^2y'' \;-\; \left(x + x^2\right)y' \;+\; 4xy \;=\; 0 \quad ; \quad r_1 = 2 \quad , \quad r_2 = 0 \quad ,$

$$y_1(x) \;=\; x^2 - \frac{2}{3}x^3 + \frac{1}{12}x^4$$

d. $x^2y'' \;+\; xy' \;+\; (4x-4)y \;=\; 0 \quad ; \quad r_1 = 2 \quad , \quad r_2 = -2 \quad ,$

$$y_1(x) = x^2\sum_{k=0}^{\infty}\frac{(-4)^k 4!}{k!(k+4)!}x^k$$

Some Answers to Some of the Exercises

WARNING! Most of the following answers were prepared hastily and late at night. They have not been properly proofread! Errors are likely!

2a. $r^2 - 3r + 2 = 0$; $r_1 = 2$, $r_2 = 1$;
$y_{\text{Euler},1}(x) = x^2$, $y_{\text{Euler},2}(x) = x$; $\lim_{x\to 0} |y_1(x)| = 0$, $\lim_{x\to 0} |y_2(x)| = 0$

2b. $r^2 - r - 2 = 0$; $r_1 = 2$, $r_2 = -1$;
$y_{\text{Euler},1}(x) = x^2$, $y_{\text{Euler},2}(x) = x^{-1}$; $\lim_{x\to 0} |y_1(x)| = 0$, $\lim_{x\to 0} |y_2(x)| = \infty$

2c. $r^2 = 0$; $r_1 = r_2 = 0$;
$y_{\text{Euler},1}(x) = 1$, $y_{\text{Euler},2}(x) = \ln|x|$; $\lim_{x\to 0} |y_1(x)| = 1$, $\lim_{x\to 0} |y_2(x)| = \infty$

2d. $2r^2 + 3r + 1 = 0$; $r_1 = -1/2$, $r_2 = -1$;
$y_{\text{Euler},1}(x) = |x|^{-1/2}$, $y_{\text{Euler},2}(x) = x^{-1}$; $\lim_{x\to 0} |y_1(x)| = \infty$, $\lim_{x\to 0} |y_2(x)| = \infty$

2e. $r^2 - 6r + 9 = 0$; $r_1 = r_2 = 3$;
$y_{\text{Euler},1}(x) = x^3$, $y_{\text{Euler},2}(x) = x^3 \ln|x|$; $\lim_{x\to 0} |y_1(x)| = 0$, $\lim_{x\to 0} |y_2(x)| = 0$

2f. $r^2 - 4 = 0$; $r_1 = 2$, $r_2 = -2$;
$y_{\text{Euler},1}(x) = x^2$, $y_{\text{Euler},2}(x) = x^{-2}$; $\lim_{x\to 0} |y_1(x)| = 0$, $\lim_{x\to 0} |y_2(x)| = \infty$

2g. $4r^2 + 4r + 1 = 0$; $r_1 = r_2 = -\frac{1}{2}$;
$y_{\text{Euler},1}(x) = |x|^{-1/2}$, $y_{\text{Euler},2}(x) = |x|^{-1/2} \ln|x|$; $\lim_{x\to 0} |y_1(x)| = \infty$, $\lim_{x\to 0} |y_2(x)| = \infty$

2h. $r^2 - 1 = 0$; $r_1 = 1$, $r_2 = -1$;
$y_{\text{Euler},1}(x) = x$, $y_{\text{Euler},2}(x) = x^{-1}$; $\lim_{x\to 0} |y_1(x)| = 0$, $\lim_{x\to 0} |y_2(x)| = \infty$

2i. $r^2 + 3r = 0$; $r_1 = 0$, $r_2 = -3$;
$y_{\text{Euler},1}(x) = 1$, $y_{\text{Euler},2}(x) = x^{-3}$; $\lim_{x\to 0} |y_1(x)| = 1$, $\lim_{x\to 0} |y_2(x)| = \infty$

2j. $r^2 - r - 6 = 0$; $r_1 = 3$, $r_2 = -2$;
$y_{\text{Euler},1}(x) = (x+2)^3$, $y_{\text{Euler},2}(x) = (x+2)^{-2}$; $\lim_{x\to 0} |y_1(x)| = 0$, $\lim_{x\to 0} |y_2(x)| = \infty$

2k. $r^2 - r = 0$; $r_1 = 1$, $r_2 = 0$;
$y_{\text{Euler},1}(x) = x - 3$, $y_{\text{Euler},2}(x) = 1$; $\lim_{x\to 3} |y_1(x)| = 0$, $\lim_{x\to 3} |y_2(x)| = 1$

2l. $2r^2 - r = 0$; $r_1 = \frac{1}{2}$, $r_2 = 0$;
$y_{\text{Euler},1}(x) = \sqrt{|x-1|}$, $y_{\text{Euler},2}(x) = 1$; $\lim_{x\to 1} |y_1(x)| = 0$, $\lim_{x\to 1} |y_2(x)| = 1$

4b. $\lim_{x\to x_0} \left|{y_1}'(x)\right| = 1$ when $r_1 = 1$; $\lim_{x\to x_0} \left|{y_1}'(x)\right| = a_1$ when $r_1 = 0$

5b. $r_1 = 1/2$, $r_2 = 0$

5e. $T_0(x) = 1$, $T_1(x) = x$, $T_2(x) = 2x^2 - 1$, $T_3(x) = 4x^3 - 3x$, $T_4(x) = 8x^4 - 8x^2 + 1$, $T_5(x) = 16x^5$, $p_5(x) = 5x - 20x^3 + 16x^5$

6a. $y_2 = y_1(x)\ln|x| \;+\; x^{3/2}\left[-2 - \frac{3}{4}x - \frac{11}{108}x^2 - \frac{25}{3{,}456}x^3 - \frac{137}{430{,}000}x^4 + \cdots\right]$

6b. $y_2 = y_1(x)\ln|x| \;+\; x\left[0 + \frac{1}{4}x + 0x^2 - \frac{3}{128}x^3 + 0x^4 + \cdots\right]$

6c. $y_2 = -6y_1(x)\ln|x| \;+\; 1 + 4x + 0x^2 - \frac{22}{3}x^3 + \frac{43}{24}x^4 + \cdots$

6d. $y_2 = -\frac{16}{9}y_1(x)\ln|x| \;+\; \frac{1}{x^2}\left[1 + \frac{4}{3}x + \frac{4}{3}x^2 + \frac{16}{9}x^3 + 0x^4 + \cdots\right]$