33

Modified Power Series Solutions and the Basic Method of Frobenius

The partial sums of a power series solution about an ordinary point x_0 of a differential equation provide fairly accurate approximations to the equation's solutions at any point x near x_0. This is true even if relatively low-order partial sums are used (provided you are just interested in the solutions at points very near x_0). However, these power series typically converge slower and slower as x moves away from x_0 towards a singular point, with more and more terms then being needed to obtain reasonably accurate partial sum approximations. As a result, the power series solutions derived in the previous two chapters usually tell us very little about the solutions near singular points. This is unfortunate because, in some applications, the behavior of the solutions near certain singular points can be a rather important issue.

Fortunately, in many of those applications, the singular point in question is not that "bad" a singular point, and a modification of the algebraic method discussed in the previous chapters can be used to obtain "modified" power series solutions about these points. That is what we will develop in this and the next two chapters.

By the way, we will only consider second-order homogeneous linear differential equations. One can extend the discussion here to first- and higher-order equations, but the important examples are all second-order.

33.1 Euler Equations and Their Solutions

The simplest examples of the sort of equations of interest in this chapter are those discussed back in chapter 19, the Euler equations. Let us quickly review them and take a look at what happens to their solutions about their singular points.

Recall that a standard second-order Euler equation is a differential equation that can be written as

$$\alpha_0 x^2 y'' + \beta_0 x y' + \gamma_0 y = 0$$

where α_0, β_0 and γ_0 are real constants with $\alpha_0 \neq 0$. Recall, also, that the basic method for solving such an equation begins with attempting a solution of the form $y = x^r$ where r is a constant to be determined. Plugging $y = x^r$ into the differential equation, we get

$$\alpha_0 x^2 \left[x^r\right]'' + \beta_0 x \left[x^r\right]' + \gamma_0 \left[x^r\right] = 0$$

$$\hookrightarrow \quad \alpha_0 x^2 \left[r(r-1)x^{r-2}\right] + \beta_0 x \left[r x^{r-1}\right] + \gamma_0 \left[x^r\right] = 0$$

$$\hookrightarrow \quad x^r \left[\alpha_0 r(r-1) + \beta_0 r + \gamma_0\right] = 0$$

$$\hookrightarrow \quad \alpha_0 r(r-1) + \beta_0 r + \gamma_0 = 0 \quad .$$

The last equation above is the indicial equation, which we typically rewrite as

$$\alpha_0 r^2 + (\beta_0 - \alpha_0) r + \gamma_0 = 0 \quad ,$$

and, from which, we can easily determine the possible values of r using basic algebra.

Generalizing slightly, we have the *shifted Euler equation*

$$\alpha_0 (x - x_0)^2 y'' + \beta_0 (x - x_0) y' + \gamma_0 y = 0$$

where x_0, α_0, β_0 and γ_0 are real constants with $\alpha_0 \neq 0$. Notice that x_0 is the one and only singular point of this equation. (Notice, also, that a standard Euler equation is a shifted Euler equation with $x_0 = 0$.)

To solve this slight generalization of a standard Euler equation, use the obvious slight generalization of the basic method for solving a standard Euler equation: First set

$$y = (x - x_0)^r \quad ,$$

where r is a yet unknown constant. Then plug this into the differential equation, compute, and simplify. Unsurprisingly, you end up with the corresponding indicial equation

$$\alpha_0 r(r-1) + \beta_0 r + \gamma_0 = 0 \quad ,$$

which you can rewrite as

$$\alpha_0 r^2 + (\beta_0 - \alpha_0) r + \gamma_0 = 0 \quad ,$$

and then solve for r, just as with the standard Euler equation. And, as with a standard Euler equation, there are then only three basic possibilities regarding the roots r_1 and r_2 to the indicial equation and the corresponding solutions to the Euler equations:

1. The two roots can be two different real numbers, $r_1 \neq r_2$.

In this case, the general solution to the differential equation is

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

with, at least when $x > x_0$,[1]

$$y_1(x) = (x - x_0)^{r_1} \quad \text{and} \quad y_2(x) = (x - x_0)^{r_2} \quad .$$

[1] In all of these cases, the formulas and observations also hold when $x < x_0$, though we may wish to replace $x - x_0$ with $|x - x_0|$ to avoid minor issues with $(x - x_0)^r$ for certain values of r (such as $r = 1/2$).

Observe that, for $j = 1$ or $j = 2$,

$$\lim_{x \to x_0^+} \left| y_j(x) \right| = \lim_{x \to x_0} |x - x_0|^{r_j} = \begin{cases} 0 & \text{if} \quad r_j > 0 \\ 1 & \text{if} \quad r_j = 0 \\ +\infty & \text{if} \quad r_j < 0 \end{cases} .$$

2. The two roots can be the same real number, $r_2 = r_1$.

   In this case, we can use reduction of order and find that the general solution to the differential equation is (when $x > x_0$)

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

with

$$y_1(x) = (x - x_0)^{r_1} \qquad \text{and} \qquad y_2(x) = (x - x_0)^{r_1} \ln |x - x_0| \quad .$$

After recalling how $\ln |X|$ behaves when $X \approx 0$ (with $X = x - x_0$), we see that

$$\lim_{x \to x_0^+} |y_2(x)| = \lim_{x \to x_0} \left| (x - x_0)^{r_1} \ln |x - x_0| \right| = \begin{cases} 0 & \text{if} \quad r_i > 0 \\ +\infty & \text{if} \quad r_i \le 0 \end{cases} .$$

3. Finally, the two roots can be complex conjugates of each other

$$r_1 = \lambda + i\omega \qquad \text{and} \qquad r_2 = \lambda - i\omega \qquad \text{with} \quad \omega > 0 \quad .$$

   After recalling that

$$X^{\lambda \pm i\omega} = X^{\lambda} \left[\cos(\omega \ln |X|) \pm i \sin(\omega \ln |X|) \right] \qquad \text{for} \quad X > 0$$

(see the discussion of complex exponents in section 19.2), we find that the general solution to the differential equation for $x > x_0$ can be given by

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

where y_1 and y_2 are the real-valued functions

$$y_1(x) = (x - x_0)^{\lambda} \cos(\omega \ln |x - x_0|)$$

and

$$y_2(x) = (x - x_0)^{\lambda} \sin(\omega \ln |x - x_0|) \quad .$$

The behavior of these solutions as $x \to x_0$ is a bit more complicated. First observe that, as X goes from 1 to 0, $\ln |X|$ goes from 0 to $-\infty$, which means that $\sin(\omega \ln |X|)$ and $\cos(\omega \ln |X|)$ then must oscillate infinitely many times between their maximum and minimum values of 1 and -1, as illustrated in figure 33.1a. So $\sin(\omega \ln |X|)$ and $\cos(\omega \ln |X|)$ are bounded, but do not approach any single value as $X \to 0$. Taking into account how X^{λ} behaves (and replacing X with $x - x_0$), we see that

$$\lim_{x \to x_0^+} |y_i(x)| = 0 \qquad \text{if} \quad \lambda > 0 \quad ,$$

and

$$\lim_{x \to x_0^+} |y_i(x)| \quad \text{does not exist if} \quad \lambda \le 0 \quad .$$

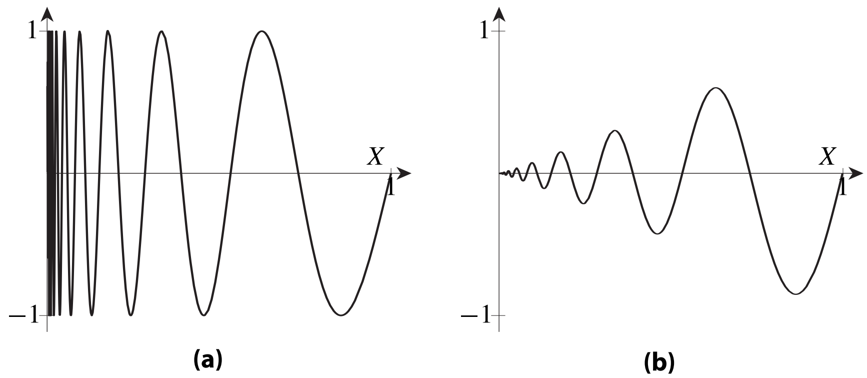

Figure 33.1: Graphs of **(a)** $\sin(\omega \ln |X|)$, and **(b)** $X^{\lambda} \sin(\omega \ln |X|)$ (with $\omega = 10$ and $\lambda = {}^{11}/_{10}$).

Notice that the behavior of these solutions as $x \rightarrow x_0$ depends strongly on the values of r_1 and r_2 . Notice, also, that you can rarely arbitrarily assign the initial values $y(x_0)$ and $y'(x_0)$ for these solutions.

!►*Example 33.1:* *Consider the shifted Euler equation*

$$(x - 3)^2 y'' - 2y = 0 \quad .$$

If $y = (x - 3)^r$ *for any constant* r *, then*

$$\begin{aligned} (x-3)^2 y'' - 2y &= 0 \\ \hookrightarrow \quad (x-3)^2 \left[(x-3)^r\right]'' - 2\left[(x-3)^r\right] &= 0 \\ \hookrightarrow \quad (x-3)^2 r(r-1)(x-3)^{r-2} - 2(x-3)^r &= 0 \\ \hookrightarrow \quad (x-3)^r \left[r(r-1) - 2\right] &= 0 \quad . \end{aligned}$$

Thus, for $y = (x-3)^r$ *to be a solution to our differential equation,* r *must satisfy the indicial equation*

$$r(r - 1) - 2 = 0 \quad .$$

After rewriting this as

$$r^2 - r - 2 = 0 \quad ,$$

and factoring,

$$(r - 2)(r + 1) = 0 \quad ,$$

we see that the two solutions to the indicial equation are $r = 2$ *and* $r = -1$ *. Thus, the general solution to our differential equation is*

$$y = c_1(x - 3)^2 + c_2(x - 3)^{-1} \quad .$$

This has one term that vanishes as $x \rightarrow 3$ *and another that blows up as* $x \rightarrow 3$ *. In particular, we cannot insist that* $y(3)$ *be any particular nonzero number, say, "*$y(3) = 2$ *".*

!►*Example 33.2:* *Now consider*

$$x^2 y'' + xy' + y = 0 \quad ,$$

which is a shifted Euler equation, but with "shift" $x_0 = 0$.

The indicial equation for this is

$$r(r-1) + r + 1 = 0 \quad ,$$

which simplifies to

$$r^2 + 1 = 0 \quad .$$

So,

$$r = \pm\sqrt{-1} = \pm i \quad .$$

In other words, $r = \lambda \pm i\omega$ with $\lambda = 0$ and $\omega = 1$. Consequently, the general solution to the differential equation in this example is given by

$$y = c_1 x^0 \cos(1 \ln|x|) + c_2 x^0 \sin(1 \ln|x|) \quad .$$

which we naturally would prefer to write more simply as

$$y = c_1 \cos(\ln|x|) + c_2 \sin(\ln|x|) \quad .$$

Here, neither term vanishes or blows up. Instead, as x goes from 1 to 0, we have $\ln|x|$ going from 0 to $-\infty$. This means that the sine and cosine terms here oscillate infinitely many times as x goes from 1 to 0, similar to the function illustrated in figure 33.1a. Again, there is no way we can require that "$y(0) = 2$".

The "blowing up" or "vanishing" of solutions illustrated above is typical behavior of solutions about the singular points of their differential equations. Sometimes it is important to know just how the solutions to a differential equation behave near a singular point x_0. For example, if you know a solution describes some process that does not blow up as $x \to x_0$, then you know that those solutions that do blow up are irrelevant to your problem (this will become very important when we study boundary-value problems).

The tools we will develop in this chapter will yield "modified power series solutions" around singular points for at least some differential equations. The behavior of the solutions near these singular points can then be deduced from these modified power series. For which equations will this analysis be appropriate? Before answering that, I must first tell you the difference between a "regular" and an "irregular" singular point.

33.2 Regular and Irregular Singular Points (and the Frobenius Radius of Convergence)

Basic Terminology

Assume x_0 is singular point on the real line for some given homogeneous linear second-order differential equation

$$ay'' + by' + cy = 0 \quad . \tag{33.1}$$

We will say that x_0 is a *regular singular point* for this differential equation if and only if that differential equation can be rewritten as

$$(x - x_0)^2 \alpha(x) y'' + (x - x_0)\beta(x) y' + \gamma(x) y = 0 \tag{33.2}$$

where α, β and γ are 'suitable, well-behaved' functions about x_0 with $\alpha(x_0) \neq 0$. The above form will be called a *quasi-Euler form about* x_0 for the given differential equation, and the shifted Euler equation

$$(x - x_0)^2 \alpha_0 y'' + (x - x_0)\beta_0 y' + \gamma_0 y = 0 \tag{33.3a}$$

where

$$\alpha_0 = \alpha(x_0) \quad , \quad \beta_0 = \beta(x_0) \quad \text{and} \quad \gamma_0 = \gamma(x_0) \quad . \tag{33.3b}$$

will be called the *associated (shifted) Euler equation (about* x_0 *)*.

Precisely what we mean above by " 'suitable, well-behaved' functions about x_0 " depends, in practice, on the coefficients of the original differential equation. In general, it means that the functions α, β and γ in equation (33.2) are all analytic at x_0. However, if the coefficients of the original equation (the a, b and c in equation (33.1)) are rational functions, then we will be able to further insist that the α, β and γ in the quasi-Euler form be polynomials.

It is quite possible that our differential equation cannot be written in quasi-Euler form about x_0. Then we will say x_0 is an *irregular singular point* for our differential equation. Thus, every singular point on the real line for our differential equation is classified as being regular or irregular depending on whether the differential equation can be rewritten in quasi-Euler form about that singular point.

While we are at it, let's also define the *Frobenious radius of analyticity* about any point z_0 for our differential equation: It is simply the distance between z_0 and the closest singular point z_s other than z_0, provided such a point exists. If no such z_s exists, then the Frobenius radius of analyticity is ∞. The Frobenius radius of analyticity will play almost the same role as played by the radius of analyticity in the previous two chapters. In fact, it is the radius of analyticity if z_0 happens to be an ordinary point.

!► ***Example 33.3 (Bessel Equations):*** *Let* ν *be any positive real constant. Bessel's equation of order* ν *is the differential equation* [2]

$$y'' + \frac{1}{x} y' + \left[1 - \left(\frac{\nu}{x}\right)^2\right] y = 0 \quad .$$

The coefficients of this,

$$a(x) = 1 \quad , \quad b(x) = \frac{1}{x} \quad \text{and} \quad c(x) = 1 - \left(\frac{\nu}{x}\right)^2 = \frac{x^2 - \nu^2}{x^2} \quad ,$$

[2] Bessel's equations and their solutions often arise in two-dimensional problems involving circular objects.

are rational functions. Multiplying through by x^2 then gives us

$$x^2 y'' + xy' + \left[x^2 - \nu^2\right] y = 0 \quad . \tag{33.4}$$

Clearly, $x_0 = 0$ is the one and only singular point for this differential equation. This, in turn, means that the Frobenius radius of analyticity about $x_0 = 0$ is ∞ (though the radius of analyticity, as defined in chapter 31, is 0 since $x_0 = 0$ is a singular point.)

Now observe that the last differential equation is

$$(x-0)^2\alpha(x)y'' + (x-0)\beta(x)y' + \gamma(x)y = 0 \quad . \tag{33.5a}$$

where α, β and γ are the simple polynomials

$$\alpha(x) = 1 \quad , \quad \beta(x) = 1 \quad \text{and} \quad \gamma(x) = x^2 - \nu^2 \quad , \tag{33.5b}$$

which are certainly analytic about $x_0 = 0$ (and every other point on the complex plane). Moreover, since

$$\alpha(0) = 1 \neq 0 \quad , \quad \beta(0) = 1 \quad \text{and} \quad \gamma(0) = -\nu^2 \quad ,$$

we now have that:

1. *The Bessel equation of order ν can be written in quasi-Euler form about $x_0 = 0$.*
2. *The singular point $x_0 = 0$ is a regular singular point.*

and

3. *The associated Euler equation about $x_0 = 0$ is*

$$(x-0)^2 \cdot 1y'' + (x-0) \cdot 1y' - \nu^2 y = 0 \quad ,$$

which, of course, is normally written

$$x^2 y'' + xy' - \nu^2 y = 0 \quad .$$

Two quick notes before going on:

1. Often, the point of interest is $x_0 = 0$, in which case we will write equation (33.2) more simply as

$$x^2\alpha(x)y'' + x\beta(x)y' + \gamma(x)y = 0 \tag{33.6}$$

with the *associated Euler equation* being

$$x^2\alpha_0 y'' + x\beta_0 y' + \gamma_0 y = 0 \tag{33.7a}$$

where

$$\alpha_0 = \alpha(0) \quad , \quad \beta_0 = \beta(0) \quad \text{and} \quad \gamma_0 = \gamma(0) \quad . \tag{33.7b}$$

2. In fact, any singular point on the complex plane can be classified as regular or irregular. However, this won't be particularly relevant to us. Our interest will only be in whether given singular points on the real line are regular or not.

Testing for Regularity

As illustrated in our last example, if the coefficients in our original differential equation are relatively simple rational functions, then it can be relatively straightforward to show that a given singular point x_0 is or is not regular by seeing if we can or cannot rewrite the equation in quasi-Euler form about x_0. An advantage of deriving this quasi-Euler form (if possible) is that we will want this quasi-Euler form in solving our differential equation. However, there are possible difficulties. If we cannot rewrite our equation in quasi-Euler form, then we may be left with the question of whether x_0 truly is an irregular singular point, or whether we just weren't clever enough to get the equation into quasi-Euler form. Also, if the coefficients in our original equation are not so simple, then the process of attempting to convert it to quasi-Euler form may be quite challenging.

A useful test for regularity is easily derived by first assuming that x_0 is a regular singular point for

$$a(x)y'' + b(x)y' + c(x)y = 0 \quad . \tag{33.8}$$

By definition, this means that this differential equation can be rewritten as

$$(x - x_0)^2\alpha(x)y'' + (x - x_0)\beta(x)y' + \gamma(x)y = 0 \tag{33.9}$$

where α, β and γ are functions analytic at x_0 with $\alpha(x_0) \neq 0$. Dividing each of these two equations by its first coefficient converts our differential equation, respectively, to the two forms

$$y'' + \frac{b(x)}{a(x)}y' + \frac{c(x)}{a(x)}y = 0$$

and

$$y'' + \frac{\beta(x)}{(x - x_0)\alpha(x)}y' + \frac{\gamma(x)}{(x - x_0)^2\alpha(x)}y = 0 \quad .$$

But these last two equations describe the same differential equation, and have the same first coefficients. Clearly, the other coefficients must also be the same, giving us

$$\frac{b(x)}{a(x)} = \frac{\beta(x)}{(x - x_0)\alpha(x)} \qquad \text{and} \qquad \frac{c(x)}{a(x)} = \frac{\gamma(x)}{(x - x_0)^2\alpha(x)} \quad .$$

Equivalently,

$$(x - x_0)\frac{b(x)}{a(x)} = \frac{\beta(x)}{\alpha(x)} \qquad \text{and} \qquad (x - x_0)^2\frac{c(x)}{a(x)} = \frac{\gamma(x)}{\alpha(x)} \quad .$$

From this and the fact that α, β and γ are analytic at x_0 with $\alpha(x_0) \neq 0$ we get that the two limits

$$\lim_{x \to x_0} (x - x_0)\frac{b(x)}{a(x)} = \lim_{x \to x_0} \frac{\beta(x)}{\alpha(x)} = \frac{\beta(x_0)}{\alpha(x_0)}$$

and

$$\lim_{x \to x_0} (x - x_0)^2\frac{c(x)}{a(x)} = \lim_{x \to x_0} \frac{\gamma(x)}{\alpha(x)} = \frac{\gamma(x_0)}{\alpha(x_0)}$$

are finite.

So, x_0 being a regular singular point assures us that the above limits are finite. Consequently, if those limits are not finite, then x_0 cannot be a regular singular point for our differential equation. That gives us

Lemma 33.1 (test for irregularity)
Assume x_0 is a singular point on the real line for

$$a(x)y'' + b(x)y' + c(x)y = 0 \quad .$$

If either of the two limits

$$\lim_{x \to x_0} (x - x_0)\frac{b(x)}{a(x)} \qquad \text{and} \qquad \lim_{x \to x_0} (x - x_0)^2 \frac{c(x)}{a(x)}$$

is not finite, then x_0 is an irregular singular point for the differential equation.

This lemma is just a test for irregularity. It can be expanded to a more complete test if we make mild restrictions on the coefficients of the original differential equation. In particular, using properties of polynomials and rational functions, we can verify the following:

Theorem 33.2 (testing for regular singular points (ver.1))
Assume x_0 is a singular point on the real line for

$$a(x)y'' + b(x)y' + c(x)y = 0$$

where a , b and c are rational functions. Then x_0 is a regular singular point for this differential equation if and only if the two limits

$$\lim_{x \to x_0} (x - x_0)\frac{b(x)}{a(x)} \qquad \text{and} \qquad \lim_{x \to x_0} (x - x_0)^2 \frac{c(x)}{a(x)}$$

are both finite values. Moreover, if x_0 is a regular singular point for the differential equation, then this differential equation can be written in quasi-Euler form

$$(x - x_0)^2 \alpha(x) y'' + (x - x_0)\beta(x) y' + \gamma(x) y = 0$$

where α , β and γ are polynomials with $\alpha(x_0) \neq 0$.

The full proof of this (along with a similar theorem applicable when the coefficients are merely quotients of analytic functions) is discussed in an appendix, section 33.7.

!►*Example 33.4:* *Consider the differential equation*

$$y'' + \frac{1}{x^2} y' + \left[1 - \frac{1}{x^2}\right] y = 0 \quad .$$

Clearly, $x_0 = 0$ is a singular point for this equation. Writing out the limits given in the above theorem (with $x_0 = 0$) yields

$$\lim_{x \to 0} (x - 0)\frac{b(x)}{a(x)} = \lim_{x \to} x \cdot \frac{x^{-2}}{1} = \lim_{x \to 0} \frac{1}{x}$$

and

$$\lim_{x \to 0} (x - 0)^2 \frac{c(x)}{a(x)} = \lim_{x \to} x^2 \cdot \frac{1 - x^{-2}}{1} = \lim_{x \to 0} \left[x^2 - 1\right] \quad ,$$

The first limit is certainly not finite. So our test for regularity tells us that $x_0 = 0$ is an irregular singular point for this differential equation.

!►Example 33.5: *Consider*

$$2xy'' - 4y' - y = 0 \quad .$$

Again, $x_0 = 0$ *is clearly the only singular point. Now,*

$$\lim_{x\to 0}(x-0)\frac{b(x)}{a(x)} = \lim_{x\to 0} x \cdot \frac{-4}{2x} = -2$$

and

$$\lim_{x\to 0}(x-0)^2\frac{c(x)}{a(x)} = \lim_{x\to 0} x^2 \cdot \frac{-1}{2x} = 0 \quad ,$$

both of which are finite. So $x_0 = 0$ *is a regular singular point for our equation, and it can be written in quasi-Euler form about* 0 *. In fact, that form is obtained by simply multiplying the original differential equation by* x *,*

$$2x^2y'' - 4xy' - xy = 0 \quad .$$

33.3 The (Basic) Method of Frobenius

Motivation and Preliminary Notes

Let's suppose we have a second-order differential equation with x_0 as a regular singular point. Then, as we just discussed, this differential equation can be written as

$$(x-x_0)^2\alpha(x)y'' + (x-x_0)\beta(x)y' + \gamma(x)y = 0$$

where α , β and γ are analytic at x_0 with $\alpha(x_0) \neq 0$. By continuity, when $x \approx x_0$, we have

$$\alpha(x) \approx \alpha_0 \quad , \quad \beta(x) \approx \beta_0 \quad \text{and} \quad \gamma(x) \approx \gamma_0$$

where

$$\alpha_0 = \alpha(x_0) \quad , \quad \beta_0 = \beta(x_0) \quad \text{and} \quad \gamma_0 = \gamma(x_0) \quad .$$

It should then seem reasonable that any solution $y = y(x)$ to

$$(x-x_0)^2\alpha(x)y'' + (x-x_0)\beta(x)y' + \gamma(x)y = 0$$

can be approximated, at least when $x \approx x_0$, by a corresponding solution to the associated shifted Euler equation

$$(x-x_0)^2\alpha_0 y'' + (x-x_0)\beta_0 y' + \gamma_0 y = 0 \quad .$$

And since some solutions to this shifted Euler equation are of the form $a_0(x-x_0)^r$ where a_0 is an arbitrary constant and r is a solution to the corresponding indicial equation,

$$\alpha_0 r(r-1) + \beta_0 r + \gamma_0 = 0 \quad ,$$

it seems reasonable to expect at least some solutions to our original differential equation to be approximated by this $a_0(x-x_0)^r$,

$$y(x) \approx a_0(x-x_0)^r \qquad \text{at least when} \qquad x \approx x_0 \quad .$$

Now, this is equivalent to saying

$$\frac{y(x)}{(x-x_0)^r} \approx a_0 \qquad \text{when} \quad x \approx x_0 \quad ,$$

which is more precisely stated as

$$\lim_{x \to x_0} \frac{y(x)}{(x-x_0)^r} = a_0 \quad . \tag{33.10}$$

At this point, there are a number of ways we might 'guess' at solutions satisfying the last approximation. Let us try a trick similar to one we've used before: Let us assume y is the known approximate solution $(x-x_0)^r$ multiplied by some yet unknown function $a(x)$,

$$y(x) = (x-x_0)^r a(x) \quad .$$

To satisfy equation (33.10), we must have

$$a(x_0) = \lim_{x \to x_0} a(x) = \lim_{x \to x_0} \frac{y(x)}{(x-x_0)^r} = a_0 \quad ,$$

telling us that $a(x)$ is reasonably well-behaved near x_0, perhaps even analytic. Well, let's hope it is analytic, because that means we can express $a(x)$ as a power series about x_0 with the arbitrary constant a_0 as the constant term,

$$a(x) = \sum_{k=0}^{\infty} a_k (x-x_0)^k \quad .$$

Then, with luck and skill, we might be able to use the methods developed in the previous chapters to find the a_k's in terms of a_0.

That is the starting point for what follows: We will assume a solution of the form

$$y(x) = (x-x_0)^r \sum_{k=0}^{\infty} a_k (x-x_0)^k$$

where r and the a_k's are constants to be determined, with a_0 being arbitrary. This will yield the "modified power series" solutions alluded to in the title of this chapter.

In the next subsection, we will describe a series of steps, generally called the *(basic) method of Frobenius*, for finding such solutions. You will discover that much of it is very similar to the algebraic method for finding power series solutions in chapter 31.

But before we start that, let me mention a few things about this method:

1. We will see that the method of Frobenius always yields at least one solution of the form

$$y(x) = (x-x_0)^r \sum_{k=0}^{\infty} a_k (x-x_0)^k$$

where r is a solution to the appropriate indicial equation. If the indicial equation for the corresponding Euler equation

$$(x-x_0)^2 \alpha_0 y'' + (x-x_0)\beta_0 y' + \gamma_0 y = 0$$

has two distinct solutions, then we will see that the method often, but not always, leads to an independent pair of such solutions.

2. If, however, that indicial equation has only one solution r, then the fact that the corresponding shifted Euler equation has a 'second solution' in the form

$$(x - x_0)^r \ln|x - x_0|$$

may lead you to suspect that a 'second solution' to our original equation is of the form

$$(x - x_0)^r \ln|x - x_0| \sum_{k=0}^{\infty} a_k (x - x_0)^k \quad .$$

That turns out to be almost, but not quite, the case.

Just what can be done when the basic Frobenius method does not yield an independent pair of solutions will be discussed in the next chapter.

The (Basic) Method of Frobenius

Suppose we wish to find the solutions about some regular singular point x_0 to some second-order homogeneous linear differential equation

$$a(x)y'' + b(x)y' + c(x)y = 0$$

with $a(x)$, $b(x)$, and $c(x)$ being rational functions.[3] In particular, let us seek solutions to *Bessel's equation of order* $^1\!/_2$,

$$y'' + \frac{1}{x}y' + \left[1 - \frac{1}{4x^2}\right]y = 0 \tag{33.11}$$

near the regular singular point $x_0 = 0$.

As with the algebraic method for finding power series solutions, there are two preliminary steps:

Pre-Step 1. If not already specified, choose the regular singular point x_0.

For our example, we choose $x_0 = 0$, which we know is the only regular singular point from the discussion in the previous section.

Pre-Step 2. Get the differential equation into the form

$$(x - x_0)^2\alpha(x)y'' + (x - x_0)\beta(x)y' + \gamma(x)y = 0 \tag{33.12}$$

where α, β and γ are polynomials, with $\alpha(x_0) \neq 0$, and with no factors shared by all three.[4]

[3] The "Frobenius method" for the more general equations is developed in chapter 35.

[4] It will actually suffice to get the differential equation into the form

$$Ay'' + By' + Cy = 0$$

where A, B and C are polynomials, preferably with no factors shared by all three. Using form (33.12) will slightly simplify bookkeeping at one point, and will be convenient for discussions later.

To get the given differential equation into the form desired, we multiply equation (33.11) by $4x^2$. That gives us the differential equation

$$4x^2y'' + 4xy' + [4x^2 - 1]y = 0 \quad . \tag{33.13}$$

(Yes, we could have just multiplied by x^2, but getting rid of any fractions will simplify computation.)

Now for the basic method of Frobenius:

Step 1. ***(a)*** Assume a solution of the form

$$y = y(x) = (x - x_0)^r \sum_{k=0}^{\infty} a_k (x - x_0)^k \tag{33.14a}$$

with a_0 being an arbitrary *nonzero* constant.[5,6]

(b) Simplify this formula for the following computations by bringing the $(x - x_0)^r$ factor into the summation,

$$y = y(x) = \sum_{k=0}^{\infty} a_k (x - x_0)^{k+r} \quad . \tag{33.14b}$$

Step 2. Compute the corresponding modified power series for y' and y'' from the assumed series for y by differentiating "term-by-term".[7] This time, you can*not* drop the "$k = 0$" term in the summations because this term is not necessarily a constant.

Since we've already decided $x_0 = 0$, we assume

$$y = y(x) = x^r \sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} a_k x^{k+r} \tag{33.15}$$

with $a_0 \neq 0$. Differentiating this term-by-term, we see that

$$\begin{aligned} y' &= \frac{d}{dx} \sum_{k=0}^{\infty} a_k x^{k+r} \\ &= \sum_{k=0}^{\infty} \frac{d}{dx}\left[a_k x^{k+r}\right] = \sum_{k=0}^{\infty} a_k (k+r) x^{k+r-1} \end{aligned}$$

and

$$\begin{aligned} y'' &= \frac{d}{dx} \sum_{k=0}^{\infty} a_k (k+r) x^{k+r-1} \\ &= \sum_{k=0}^{\infty} \frac{d}{dx}\left[a_k (k+r) x^{k+r-1}\right] = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1) x^{k+r-2} \quad . \end{aligned}$$

[5] Insisting that $a_0 \neq 0$ will be important in determining the possible values of r.

[6] This procedure is valid whether or not $x - x_0$ is positive or negative. However, a few readers may have concerns about $(x - x_0)^r$ being imaginary if $x < x_0$ and r is, say, $1/2$. If you are one of those readers, go ahead and assume $x > x_0$ for now, and later read the short discussion of solutions on intervals with $x < x_0$ on page 33–23.

[7] If you have any qualms about "term-by-term" differentiation here, see exercise 33.6 at the end of the chapter.

Step 3. Plug these series for y, y', and y'' back into the differential equation, "multiply things out", and divide out the $(x - x_0)^r$ to get the left side of your equation in the form of the sum of a few power series about x_0.[8,9]

Combining the above series formulas for y, y' and y'' with our differential equation (equation (33.13)), we get

$$\begin{aligned}
0 &= 4x^2 y'' + 4xy' + [4x^2 - 1]y \\
&= 4x^2 \sum_{k=0}^{\infty} a_k (k+r)(k+r-1) x^{k+r-2} + 4x \sum_{k=0}^{\infty} a_k (k+r) x^{k+r-1} \\
&\quad + [4x^2 - 1] \sum_{k=0}^{\infty} a_k x^{k+r} \\
&= 4x^2 \sum_{k=0}^{\infty} a_k (k+r)(k+r-1) x^{k+r-2} + 4x \sum_{k=0}^{\infty} a_k (k+r) x^{k+r-1} \\
&\quad + 4x^2 \sum_{k=0}^{\infty} a_k x^{k+r} - 1 \sum_{k=0}^{\infty} a_k x^{k+r} \\
&= \sum_{k=0}^{\infty} a_k 4(k+r)(k+r-1) x^{k+r} + \sum_{k=0}^{\infty} a_k 4(k+r) x^{k+r} \\
&\quad + \sum_{k=0}^{\infty} a_k 4 x^{k+2+r} + \sum_{k=0}^{\infty} a_k (-1) x^{k+r} \quad .
\end{aligned}$$

Dividing out the x^r from each term then yields

$$\begin{aligned}
0 = \sum_{k=0}^{\infty} a_k 4(k+r)(k+r-1) x^{k} &+ \sum_{k=0}^{\infty} a_k 4(k+r) x^{k} \\
&+ \sum_{k=0}^{\infty} a_k 4 x^{k+2} + \sum_{k=0}^{\infty} a_k (-1) x^{k} \quad .
\end{aligned}$$

Step 4. For each series in your last equation, do a change of index so that each series looks like

$$\sum_{n=\text{something}}^{\infty} \big[\text{something not involving } x\big](x - x_0)^n \quad .$$

Be sure to appropriately adjust the lower limit in each series.

In all but the third series in the example, the change of index is trivial, $n = k$. In the third series, we will set $n = k + 2$ (equivalently, $n - 2 = k$). This

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[8] Dividing out the $(x - x_0)^r$ isn't necessary, but it simplifies the expressions slightly and reduces the chances of silly errors later.

[9] You may want to turn your paper sideways for more room!

means, in the third series, replacing k with $n-2$, and replacing $k=0$ with $n = 0+2 = 2$:

$$0 = \underbrace{\sum_{k=0}^{\infty} a_k 4(k+r)(k+r-1)x^k}_{n=k} + \underbrace{\sum_{k=0}^{\infty} a_k 4(k+r)x^k}_{n=k} + \underbrace{\sum_{k=0}^{\infty} a_k 4x^{k+2}}_{n=k+2} + \underbrace{\sum_{k=0}^{\infty} a_k(-1)x^k}_{n=k}$$

$$= \sum_{n=0}^{\infty} a_n 4(n+r)(n+r-1)x^n + \sum_{n=0}^{\infty} a_n 4(n+r)x^n + \sum_{n=2}^{\infty} a_{n-2} 4x^n + \sum_{n=0}^{\infty} a_n(-1)x^n \quad .$$

Step 5. Convert the sum of series in your last equation into one big series. The first few terms will probably have to be written separately. Simplify what can be simplified.

Since one of the series in the last equation begins with $n = 2$, we need to separate out the terms corresponding to $n = 0$ and $n = 1$ in the other series before combining the series:

$$0 = \sum_{n=0}^{\infty} a_n 4(n+r)(n+r-1)x^n + \sum_{n=0}^{\infty} a_n 4(n+r)x^n + \sum_{n=2}^{\infty} a_{n-2} 4x^n + \sum_{n=0}^{\infty} a_n(-1)x^n$$

$$= \left[a_0 \underbrace{4(0+r)(0+r-1)}_{4r(r-1)} x^0 + a_1 \underbrace{4(1+r)(1+r-1)}_{4(1+r)r} x^1 + \sum_{n=2}^{\infty} a_n 4(n+r)(n+r-1)x^n \right]$$
$$+ \left[a_0 \underbrace{4(0+r)}_{4r} x^0 + a_1 \underbrace{4(1+r)}_{4(1+r)} x^1 + \sum_{n=2}^{\infty} a_n 4(n+r)x^n \right]$$
$$+ \sum_{n=2}^{\infty} a_{n-2} 4x^n + \left[-a_0 x^0 - a_1 x^1 + \sum_{n=2}^{\infty} a_n(-1)x^n \right]$$

$$= a_0 \big[\underbrace{4r(r-1) + 4r - 1}_{4r^2 - 4r + 4r - 1} \big] x^0 + a_1 \big[\underbrace{4(1+r)r + 4(1+r) - 1}_{4r + 4r^2 + 4 + 4r - 1} \big] x^1$$
$$+ \sum_{n=2}^{\infty} \big[\underbrace{a_n 4(n+r)(n+r-1) + a_n 4(n+r) + a_{n-2} 4 + a_n(-1)}_{a_n[4(n+r)(n+r-1) + 4(n+r) - 1] + 4a_{n-2}} \big] x^n$$

$$= a_0\big[4r^2 - 1\big]x^0 + a_1\big[4r^2 + 8r + 3\big]x^1 + \sum_{n=2}^{\infty}\Big[a_n\big[4(n+r)(n+r) - 1\big] + 4a_{n-2}\Big]x^n \quad .$$

So our differential equation reduces to

$$a_0\big[4r^2 - 1\big]x^0 + a_1\big[4r^2 + 8r + 3\big]x^1 + \sum_{n=2}^{\infty}\Big[a_n\big[4(n+r)^2 - 1\big] + 4a_{n-2}\Big]x^n = 0 \quad . \tag{33.16}$$

Step 6. The first term in the last equation just derived will be of the form

$$a_0\big[\text{formula of } r\big](x - x_0)^0 \quad .$$

Since each term in the series must vanish, we must have

$$a_0\big[\text{formula of } r\big] = 0 \quad .$$

Moreover, since $a_0 \neq 0$ (by assumption), the above must reduce to

$$\text{formula of } r = 0 \quad .$$

This is the *indicial equation* for the differential equation. It will be a quadratic equation (we'll see why later). Solve this equation for r. You will get two solutions (sometimes called either the *exponents* of the solution or the *exponents* of the singularity). Denote them by r_1 and r_2. If the exponents are real (which is common in applications), label the exponents so that $r_1 \geq r_2$. If the exponents are not real, then it does not matter which is labeled as r_1.[10]

In our example, the first term in the "big series" is the first term in equation (33.16),

$$a_0\big[4r^2 - 1\big]x^0 \quad .$$

Since this must be zero (and $a_0 \neq 0$ by assumption) the indicial equation is

$$4r^2 - 1 = 0 \quad . \tag{33.17}$$

Thus,

$$r = \pm\sqrt{\frac{1}{4}} = \pm\frac{1}{2} \quad .$$

Following the convention given above (that $r_1 \geq r_2$),

$$r_1 = \frac{1}{2} \quad \text{and} \quad r_2 = -\frac{1}{2} \quad .$$

[10] We are assuming the coefficients of our differential equation— α, β and γ — are real-valued functions on the real line. In the very unlikely case they are not, then a more general convention should be used: If the solutions to the indicial equation differ by an integer, then label them so that $r_1 - r_2 \geq 0$. Otherwise, it does not matter which you call r_1 and which you call r_2. The reason for this convention will become apparent later (in section 33.5) after we further discuss the formulas arising from the Frobenious method.

Step 7. Using r_1 (the *largest* r if the exponents are real):

(a) Plug r_1 into the last series equation (and simplify, if possible). This will give you an equation of the form

$$\sum_{n=n_0}^{\infty} \left[n^{\text{th}} \text{ formula of } a_j\text{'s} \right] (x - x_0)^n = 0 \quad .$$

Since each term must vanish, we must have

$$n^{\text{th}} \text{ formula of } a_j\text{'s} = 0 \qquad \text{for} \quad n_0 \leq n \quad .$$

(b) Solve this last set of equations for

$$a_{\text{highest index}} = \text{formula of } n \text{ and lower indexed } a_j\text{'s} \quad .$$

A few of these equations may need to be treated separately, but you will also obtain a relatively simple formula that holds for all indices above some fixed value. This formula is the *recursion formula* for computing each coefficient a_n from the previously computed coefficients.

(c) (Optional) To simplify things just a little, do another change of indices so that the recursion formula just derived is rewritten as

$$a_k = \text{formula of } k \text{ and lower-indexed coefficients} \quad .$$

Letting $r = r_1 = 1/2$ *in equation (33.16) yields*

$$\begin{aligned}
0 &= a_0\left[4r^2 - 1\right]x^0 + a_1\left[4r^2 + 8r + 3\right]x^1 \\
&\quad + \sum_{n=2}^{\infty}\left[a_n\left[4(n+r)^2 - 1\right] + 4a_{n-2}\right]x^n \\
&= a_0\left[4\left(\tfrac{1}{2}\right)^2 - 1\right]x^0 + a_1\left[4\left(\tfrac{1}{2}\right)^2 + 8\left(\tfrac{1}{2}\right) + 3\right]x^1 \\
&\quad + \sum_{n=2}^{\infty}\left[a_n\left[4\left(n + \tfrac{1}{2}\right)^2 - 1\right] + 4a_{n-2}\right]x^n \\
&= a_0 0x^0 + a_1 8x^1 + \sum_{n=2}^{\infty}\left[a_n\left[4n^2 + 4n\right] + 4a_{n-2}\right]x^n \quad .
\end{aligned}$$

The first term vanishes (as it should since $r = 1/2$ *satisfies the indicial equation, which came from making the first term vanish). Doing a little more simple algebra, we see that, with* $r = 1/2$*, equation (33.16) reduces to*

$$0a_0x^0 + 8a_1x^1 + \sum_{n=2}^{\infty} 4\left[n(n+1)a_n + a_{n-2}\right]x^n = 0 \quad . \qquad (33.18)$$

Since the individual terms in this series must vanish, we have

$$0a_0 = 0 \quad , \quad 8a_1 = 0$$

and

$$n(n+1)a_n + a_{n-2} = 0 \quad \text{for} \quad n = 2, 3, 4 \ldots \quad .$$

Solving for a_n *gives us the recursion formula*

$$a_n = \frac{-1}{n(n+1)} a_{n-2} \quad \text{for} \quad n = 2, 3, 4 \ldots \quad .$$

Using the trivial change of index, $k = n$ *, this is*

$$a_k = \frac{-1}{k(k+1)} a_{k-2} \quad \text{for} \quad k = 2, 3, 4 \ldots \quad . \tag{33.19}$$

(d) Use the recursion formula (and any corresponding formulas for the lower-order terms) to find all the a_k's in terms of a_0 and, possibly, one other a_m. Look for patterns!

From the first two terms in equation (33.18),

$$0a_0 = 0 \quad \Longrightarrow \quad a_0 \text{ is arbitrary.}$$

$$8a_1 = 0 \quad \Longrightarrow \quad a_1 = 0 \quad .$$

Using these values and recursion formula (33.19) with $k = 2, 3, 4, \ldots$ *(and looking for patterns):*

$$a_2 = \frac{-1}{2(2+1)} a_{2-2} = \frac{-1}{2 \cdot 3} a_0 \quad ,$$

$$a_3 = \frac{-1}{3(3+1)} a_{3-2} = \frac{-1}{3 \cdot 4} a_1 = \frac{-1}{3 \cdot 4} \cdot 0 = 0 \quad ,$$

$$a_4 = \frac{-1}{4(4+1)} a_{4-2} = \frac{-1}{4 \cdot 5} a_2 = \frac{-1}{4 \cdot 5} \cdot \frac{-1}{2 \cdot 3} a_0 = \frac{(-1)^2}{5 \cdot 4 \cdot 3 \cdot 2} a_0 = \frac{(-1)^2}{5!} a_0 \quad ,$$

$$a_5 = \frac{-1}{5(5+1)} a_{5-2} = \frac{-1}{5 \cdot 6} \cdot 0 = 0 \quad ,$$

$$a_6 = \frac{-1}{6(6+1)} a_{6-2} = \frac{-1}{6 \cdot 7} a_4 = \frac{-1}{7 \cdot 6} \cdot \frac{(-1)^2}{5!} a_0 = \frac{(-1)^3}{7!} a_0 \quad ,$$

$$\vdots$$

The patterns should be obvious here:

$$a_k = 0 \quad \text{for} \quad k = 1, 3, 5, 7, \ldots \quad ,$$

and

$$a_k = \frac{(-1)^{k/2}}{(k+1)!} a_0 \quad \text{for} \quad k = 2, 4, 6, 8, \ldots \quad .$$

Using $k = 2m$ *, this can be written more conveniently as*

$$a_{2m} = (-1)^m \frac{a_0}{(2m+1)!} \quad \text{for} \quad m = 1, 2, 3, 4, \ldots \quad .$$

Moreover, this last equation reduces to the trivially true statement "$a_0 = a_0$" if $m = 0$. So, in fact, it gives all the even-indexed coefficients,

$$a_{2m} = (-1)^m \frac{a_0}{(2m+1)!} \qquad \text{for} \quad m = 0,\ 1,\ 2,\ 3,\ 4,\ \ldots \quad .$$

(e) Using $r = r_1$ along with the formulas just derived for the coefficients, write out the resulting series for y. Try to simplify it and factor out the arbitrary constant(s).

Plugging $r = 1/2$ and the formulas just derived for the a_k's into the formula originally assumed for y (equation (33.15) on page 33–13), we get

$$\begin{aligned} y &= x^r \sum_{k=0}^{\infty} a_k x^k \\ &= x^r \left[\sum_{\substack{k=0 \\ k \text{ odd}}}^{\infty} a_k x^k + \sum_{\substack{k=0 \\ k \text{ even}}}^{\infty} a_k x^k \right] \\ &= x^{1/2} \left[\sum_{\substack{k=0 \\ k \text{ odd}}}^{\infty} 0 \cdot x^k + \sum_{m=0}^{\infty} (-1)^m \frac{a_0}{(2m+1)!} x^{2m} \right] \\ &= x^{1/2} \left[0 + a_0 \sum_{m=0}^{\infty} (-1)^m \frac{1}{(2m+1)!} x^{2m} \right] \quad . \end{aligned}$$

So one set of solutions to Bessel's equation of order $1/2$ (equation (33.11) on page 33–12) is given by

$$y = a_0 x^{1/2} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} x^{2m} \tag{33.20}$$

with a_0 being an arbitrary constant.

Step 8. If the indical equation has two distinct solutions, now repeat step 7 with the other exponent, r_2, replacing r_1.[11] If $r_2 = r_1$, just go to the next step.

Sometimes (but not always) this step will lead to a solution of the form

$$(x - x_0)^{r_2} \sum_{k=0}^{\infty} a_k (x - x_0)^k \quad .$$

with a_0 being the only arbitrary constant. But sometimes this step leads to a series having two arbitrary constants, with one being multiplied by the series solution already found in the previous steps (our example will illustrate this). And sometimes this step leads to a contradiction, telling you that there is no solution of the form

$$(x - x_0)^{r_2} \sum_{k=0}^{\infty} a_k (x - x_0)^k \qquad \text{with} \quad a_0 \neq 0 \quad .$$

(We'll discuss this further in *Problems Possibly Arising in Step 8* starting on page 33–28.)

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[11] But first see *Dealing with Complex Exponents* starting on page 33–32 if the exponents are complex.

Letting $r = r_2 = -1/2$ in equation (33.16) yields

$$\begin{aligned} 0 &= a_0\big[4r^2 - 1\big]x^0 + a_1\big[4r^2 + 8r + 3\big]x^1 \\ &\quad + \sum_{n=2}^{\infty}\Big[a_n\big[4(n+r)^2 - 1\big] + 4a_{n-2}\Big]x^n \\ &= a_0\left[4\left(-\frac{1}{2}\right)^2 - 1\right]x^0 + a_1\left[4\left(-\frac{1}{2}\right)^2 + 8\left(-\frac{1}{2}\right) + 3\right]x^1 \\ &\quad + \sum_{n=2}^{\infty}\left[a_n\left[4\left(n - \frac{1}{2}\right)^2 - 1\right] + 4a_{n-2}\right]x^n \\ &= a_0 0x^0 + a_1 0x^1 + \sum_{n=2}^{\infty}\left[a_n\left[4n^2 - 4n\right] + 4a_{n-2}\right]x^n \end{aligned}$$

That is,

$$0a_0x^0 + 0a_1x^1 + \sum_{n=2}^{\infty} 4\left[a_n n(n-1) + a_{n-2}\right]x^n = 0 \quad ,$$

which means that

$$0a_0 = 0 \quad , \quad 0a_1 = 0$$

and

$$a_n n(n-1) + a_{n-2} = 0 \quad \text{for} \quad n = 2, 3, 4, \ldots \quad .$$

This tells us that a_0 and a_1 can be any values (i.e., are arbitrary constants) and that the remaining a_n's can be computed from these two arbitrary constants via the recursion formula

$$a_n = \frac{-1}{n(n-1)}a_{n-2} \quad \text{for} \quad n = 2, 3, 4, \ldots \quad .$$

That is,

$$a_k = \frac{-1}{k(k-1)}a_{k-2} \quad \text{for} \quad k = 2, 3, 4, \ldots \quad . \tag{33.21}$$

Why don't you finish these computations as an exercise? You should have no trouble in obtaining

$$y = a_0 x^{-1/2}\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m)!}x^{2m} + a_1 x^{-1/2}\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)!}x^{2m+1} \quad . \tag{33.22}$$

Note that the second series term is the same series (slightly rewritten) as in equation (33.20) (since $x^{-1/2}x^{2m+1} = x^{1/2}x^{2m}$).

?►Exercise 33.1: *Do the computations left "as an exercise" in the last statement.*

Step 9. If the last step yielded y as an arbitrary linear combination of two different series, then that is the general solution to the original differential equation. If the last step yielded y as just one arbitrary constant times a series, then the general solution to the original differential equation is the linear combination of the two series obtained at the end of steps 7 and 8. Either way, write down the general solution (using different symbols for the two different arbitrary constants!). If step 8 did not yield a new series solution, then at least write down the one solution previously derived, noting that a second solution is still needed for the general solution to the differential equation.

We are in luck. In the last step we obtained y as the linear combination of two different series (equation (33.22)). So

$$y = a_0 x^{-1/2} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!} x^{2m} + a_1 x^{-1/2} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} x^{2m+1} \quad ,$$

is the general solution to our original differential equation (equation (33.11) — Bessel's equation of order $1/2$).

Last Step. See if you recognize the series as the series for some well-known function (you probably won't!).

Our luck continues! The two power series in our last formula for y are easily recognized as the Taylor series for the cosine and sine functions,

$$\begin{aligned} y &= a_0 x^{-1/2} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!} x^{2m} + a_1 x^{-1/2} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} x^{2m+1} \\ &= a_0 x^{-1/2} \cos(x) + a_1 x^{-1/2} \sin(x) \quad . \end{aligned}$$

So we can also write the general solution to Bessel's equation of order $1/2$ as

$$y = a_0 \frac{\cos(x)}{\sqrt{x}} + a_1 \frac{\sin(x)}{\sqrt{x}} \quad . \tag{33.23}$$

"First" and "Second" Solutions

For purposes of discussion, it is convenient to refer to the first solution found in the basic method of Frobenius,

$$y(x) = (x - x_0)^{r_1} \sum_{k=0}^{\infty} a_k (x - x_0)^k \quad ,$$

as the *first solution*. If we then pick a particular nonzero value for a_0, then we have a *first particular solution*. In the above, our first solution was

$$y(x) = a_0 \, x^{1/2} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} x^{2m} \quad ,$$

Taking $a_0 = 1$, we then have a first particular solution

$$y_1(x) = x^{1/2} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} x^{2m} \quad .$$

Naturally, if we also obtain a solution corresponding to r_2 in step 8,

$$y(x) = (x - x_0)^{r_2} \sum_{k=0}^{\infty} a_k (x - x_0)^k \quad ,$$

then we will refer to that as our *second solution*, with a *second particular solution* being this with a specific nonzero value chosen for a_0 and any specific value chosen for any other arbitrary constant. For example, in illustrating the basic method of Frobenius, we obtained

$$y(x) = a_0 x^{-1/2} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!} x^{2m} + a_1 x^{-1/2} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} x^{2m+1}$$

as our second solution. Choosing $a_0 = 1$ and $a_1 = 0$, we get the second particular solution

$$y_2(x) = x^{-1/2} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!} x^{2m} \quad .$$

While we are at it, let us agree that, for the rest of this chapter, x_0 always denotes a regular singular point for some differential equation of interest, and that r_1 and r_2 always denote the corresponding exponents; that is, the solutions to the corresponding indicial equation. Let us further agree that these exponents are indexed according to the convention given in the basic Frobenius, with $r_1 \geq r_2$ if they are real.

33.4 Basic Notes on Using the Frobenius Method

The Obvious

One thing should be obvious: The method we've just outlined is even longer and more tedious than the algebraic method used in chapter 31 to find power series solutions to similar equations about ordinary points. On the other hand, much of this method is based on that algebraic method, which, by now, you have surely mastered.

Naturally, all the 'practical advice' given regarding the algebraic method in chapter 31 still holds, including the recommendation that you use

$$Y(X) = y(x) \qquad \text{with} \quad X = x - x_0$$

to simplify your calculations when $x_0 \neq 0$.

But there are a number of other things you should be aware of:

Solutions on Intervals with $x < x_0$

On an interval with $x < x_0$, $(x - x_0)^r$ might be complex-valued (or even ambiguous) if r is not an integer. For example,

$$(x - x_0)^{1/2} = (-|x - x_0|)^{1/2} = (-1)^{1/2} |x - x_0|^{1/2} = \pm i\, |x - x_0|^{1/2} \quad .$$

More generally, we will have

$$(x - x_0)^r = (-|x - x_0|)^r = (-1)^r |x - x_0|^r \qquad \text{when} \quad x < x_0 \quad .$$

But this is not a significant issue because $(-1)^r$ can be viewed as a constant (possibly complex) and can be divided out of the final formula (or incorporated in the arbitrary constants). Thus, in our final formulas for $y(x)$, we can replace

$$(x - x_0)^r \quad \text{with} \quad |x - x_0|^r \quad .$$

to avoid having any explicitly complex-valued expressions (at least when r is not complex). Let's keep in mind that this is only needed if r is not an integer.

Convergence of the Series

It probably won't surprise you to learn that the Frobenius radius of analyticity serves as a lower bound on the radius of convergence for the power series found in the Frobenius method. To be precise, we have the following theorem:

Theorem 33.3
Let x_0 be a regular singular point for some second-order homogeneous linear differential equation

$$a(x)y'' + b(x)y' + c(x)y = 0$$

and let

$$y(x) = |x - x_0|^r \sum_{k=0}^{\infty} a_k (x - x_0)^k$$

be a modified power series solution to the differential equation found by the basic method of Frobenius. Then the radius of convergence for $\sum_{k=0}^{\infty} a_k (x - x_0)^k$ is at least equal to the Frobenius radius of convergence about x_0 for this differential equation.

We will verify this claim in chapter 35. For now, let us simply note that this theorem assures us that the given solution y is valid at least on the intervals

$$(x_0 - R, x_0) \qquad \text{and} \qquad (x_0, x_0 + R)$$

where R is that Frobenious radius of convergence. Whether or not we can include the point x_0 depends on the value of the exponent r.

!►*Example 33.6:* *To illustrate the Frobenius method, we found modified power series solutions about $x_0 = 0$*

$$y_1(x) = x^{1/2} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} x^{2m} \qquad \text{and} \qquad y_2(x) = x^{-1/2} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!} x^{2m}$$

for Bessel's equation of order $^1\!/_2$,

$$y'' + \frac{1}{x}y' + \left[1 - \frac{1}{4x^2}\right]y = 0 \quad .$$

Since there are no singular points for this differential equation other than $x_0 = 0$ *, the Frobenius radius of convergence for this differential equation about* $x_0 = 0$ *is* $R = \infty$ *. That means the power series in the above formulas for* y_1 *and* y_2 *converge everywhere.*

However, the $x^{1/2}$ *and* $x^{-1/2}$ *factors multiplying these power series are not "well behaved" at* $x_0 = 0$ *— neither is differentiable there, and one becomes infinite as* $x \to 0$ *. So, the above formulas for* y_1 *and* y_2 *are valid only on intervals not containing* $x = 0$ *, the largest of which are* $(0, \infty)$ *and* $(-\infty, 0)$ *. Of course, on* $(-\infty, 0)$ *the square roots yield imaginary values, and we would prefer using the solutions*

$$y_1(x) = |x|^{1/2} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} x^{2m} \qquad \text{and} \qquad y_2(x) = |x|^{-1/2} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!} x^{2m} \quad .$$

Variations of the Method

Naturally, there are several variations of "the basic method of Frobenius". The one just given is merely one the author finds convenient for initial discussion.

One variation you may want to consider is to find particular solutions without arbitrary constants by setting " $a_0 = 1$ ". That is, in step 1, assume

$$y(x) = (x - x_0)^r \sum_{k=0}^{\infty} a_k (x - x_0)^k \qquad \text{with} \quad a_0 = 1 \quad .$$

Assuming a_0 is 1 , instead of an arbitrary nonzero constant, leads to a first particular solution

$$y_1(x) = (x - x_0)^{r_1} \sum_{k=0}^{\infty} a_k (x - x_0)^k \qquad \text{with} \quad a_0 = 1 \quad .$$

With a little thought, you will realize that this is exactly the same as you would have obtained at the end of step 7, only not multiplied by an arbitrary constant. In particular, had we done this with the Bessel's equation used to illustrate the method, we would have obtained

$$y_1(x) = x^{1/2} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} x^{2m}$$

instead of formula (33.20) on page 33–19.

If $r_2 \neq r_1$, then, with luck, doing step 8 will yield a second particular solution

$$y_2(x) = (x - x_0)^{r_2} \sum_{k=0}^{\infty} a_k (x - x_0)^k \qquad \text{with} \quad a_0 = 1 \quad .$$

As mentioned, one of the a_k's other than a_0 may also be arbitrary. Set that equal to your favorite number for computation, 0 . In particular, doing this with the illustrating example would have yielded

$$y_2(x) = x^{-1/2} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!} x^{2m} \quad ,$$

instead of formula (33.22) on page 33–20.

Assuming the second particular solution can be found, this variant of the method yields a pair of particular solutions $\{y_1, y_2\}$ that, because $r_1 \neq r_1$, is easily seen to be linearly independent over any interval on which the formulas are valid. Thus,

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

is the general solution to the differential equation over this interval.

Using the Method When x_0 is an Ordinary Point

It should be noted that we can also find the power series solutions of a differential equation about an ordinary point x_0 using the basic Frobenius method (ignoring, of course, the first preliminary step). In practice, though, it would be silly to go through the extra work in the Frobenius method when you can use the shorter algebraic method from chapter 31. After all, if x_0 is an ordinary point for our differential equation, then we already know the solution y can be written as

$$y(x) = a_0 y_1(x) + a_1 y_2(x)$$

where $y_1(x)$ and $y_2(x)$ are power series about x_0 with

$$\begin{aligned} y_1(x) &= 1 + \text{a summation of terms of order } 2 \text{ or more} \\ &= (x - x_0)^0 \sum_{k=0}^{\infty} b_k (x - x_0)^k \qquad \text{with} \quad b_k = 1 \end{aligned}$$

and

$$\begin{aligned} y_2(x) &= 1 \cdot (x - x_0) + \text{a summation of terms of order } 2 \text{ or more} \\ &= (x - x_0)^1 \sum_{k=0}^{\infty} c_k (x - x_0)^k \qquad \text{with} \quad c_k = 1 \end{aligned}$$

(see *Initial-Value Problems* on page 31–24). Clearly then, $r_2 = 0$ and $r_1 = 1$, and the indicial equation that would arise in using the Frobenius method would just be $r(r-1) = 0$.

So why bother solving for the exponents r_1 and r_2 of a differential equation at a point x_0 when you don't need to? It's easy enough to determine whether a point x_0 is an ordinary point or a regular singular point for your differential equation. Do so, and don't waste your time using the Frobenius method unless the point in question is a regular singular point.

33.5 About the Indicial and Recursion Formulas

In chapter 35, we will closely examine the formulas involved in the basic Frobenius method. Here are a few things regarding the indicial equation and the recursion formulas that we will verify then (and which you should observe in your own computations now).

The Indicial Equation and the Exponents

Remember that one of the preliminary steps has us rewriting the differential equation as

$$(x - x_0)^2\alpha(x)y'' + (x - x_0)\beta(x)y' + \gamma(x)y = 0$$

where α, β and γ are polynomials, with $\alpha(x_0) \neq 0$, and with no factors shared by all three. In practice, these polynomials are almost always real (i.e., their coefficients are real numbers). Let us assume this.

If you carefully follow the subsequent computations in the basic Frobenius method, you will discover that the indicial equation is just as we suspected at the start of section 33.3; namely,

$$\alpha_0 r(r-1) + \beta_0 r + \gamma_0 = 0 \quad ,$$

where

$$\alpha_0 = \alpha(x_0) \quad , \quad \beta_0 = \beta(x_0) \quad \text{and} \quad \gamma_0 = \gamma(x_0) \quad .$$

The exponents for our differential equation (i.e., the solutions to the indicial equation) can then be found by rewriting the indicial equation as

$$\alpha_0 r^2 + (\beta_0 - \alpha_0)r + \gamma_0 = 0$$

and using basic algebra.

!►*Example 33.7:* *In describing the Frobenius method, we used Bessel's equation of order* $^1/_2$ *with* $x_0 = 0$. *That equation was (on page 33–13) rewritten as*

$$4x^2y'' + 4xy' + [4x^2 - 1]y = 0 \quad ,$$

which is

$$(x - 0)^2\alpha(x)y'' + (x - 0)\beta(x)y' + \gamma(x)y = 0$$

with

$$\alpha(x) = 4 \quad , \quad \beta(x) = 4 \quad \text{and} \quad \gamma(x) = 4x^2 - 1 \quad .$$

So

$$\alpha_0 = \alpha(0) = 4 \quad , \quad \beta_0 = \beta(0) = 4 \quad \text{and} \quad \gamma_0 = \gamma(0) = -1 \quad .$$

According to the above, the corresponding indicial equation should be

$$4r(r-1) + 4r - 1 = 0 \quad ,$$

which simplifies to

$$4r^2 - 1 = 0 \quad ,$$

and which, indeed, is what we obtained (equation (33.17) on page 33–16) as the indicial equation for Bessel's equation of order $^1/_2$.

From basic algebra, we know that, if the coefficients of the indicial equation

$$\alpha_0 r^2 + (\beta_0 - \alpha_0)r + \gamma_0 = 0$$

are all real numbers, then the two solutions to the indicial equation are either both real numbers (possibly the same real numbers) or are complex conjugates of each other. This is usually the case in practice (it is always the case in the examples and exercises of this chapter).

The Recursion Formulas

After finding the solutions r_1 and r_2 to the indicial equation, the basic method of Frobenius has us deriving the recursion formula corresponding to each of these r's. In chapter 35, we'll discover that each of these recursion formulas can always be written as

$$a_k = \frac{F(k, r, a_0, a_1, \ldots, a_{k-1})}{(k + r - r_1)(k + r - r_2)} \quad \text{for} \quad k \geq \kappa \tag{33.24a}$$

where κ is some positive integer, and r is either r_1 or r_2, depending on whether this is the recursion formula corresponding to r_1 or r_2, respectively. It also turns out that this formula is derived from the seemingly equivalent relation

$$(k + r - r_1)(k + r - r_2)\, a_k = F(k, r, a_0, a_1, \ldots, a_{k-1}) \quad , \tag{33.24b}$$

which holds for every integer k greater than or equal to κ. (This later equation will be useful when we discuss possible "degeneracies" in the recursion formula).

At this point, all you need to know about $F(k, r, a_0, a_1, \ldots, a_{k-1})$ is that it is a formula that yields a finite value for every possible choice of its variables. Unfortunately, it is not that easily described, nor would knowing it be all that useful at this point.

Of greater interest is the fact that the denominator in the recursion formula factors so simply and predicably. The recursion formula obtained using $r = r_1$ will be

$$a_k = \frac{F(k, r_1, a_0, a_1, \ldots, a_{k-1})}{(k + r_1 - r_1)(k + r_1 - r_2)} = \frac{F(k, r_1, a_0, a_1, \ldots, a_{k-1})}{k(k + r_1 - r_2)} \quad ,$$

while the recursion formula obtained using $r = r_2$ will be

$$a_k = \frac{F(k, r_2, a_0, a_1, \ldots, a_{k-1})}{(k + r_2 - r_1)(k + r_2 - r_2)} = \frac{F(k, r_2, a_0, a_1, \ldots, a_{k-1})}{(k - [r_1 - r_2])k} \quad .$$

!►*Example 33.8:* *In solving Bessel's equation of order* $^1\!/_2$ *we obtained*

$$r_1 = \frac{1}{2} \quad \text{and} \quad r_2 = -\frac{1}{2} \quad .$$

The recursion formulas corresponding to each of these were found to be

$$a_k = \frac{-1}{k(k+1)} a_{k-2} \quad \text{and} \quad a_k = \frac{-1}{k(k-1)} a_{k-2} \quad \text{for} \quad k = 2,\ 3,\ 4,\ \ldots \quad ,$$

respectively (see formulas (33.19) and (33.21)). Looking at the recursion formula corresponding to $r = {}^1\!/_2$ *we see that, indeed,*

$$a_k = \frac{-1}{k(k+1)} a_{k-2} = \frac{-1}{k\left(k + \frac{1}{2} - \left[-\frac{1}{2}\right]\right)} a_{k-2} = \frac{-1}{k\,(k + r_1 - r_2)} a_{k-2} \quad .$$

Likewise, looking at the recursion formula corresponding to $r = {}^1\!/_2$ *we see that*

$$a_k = \frac{-1}{(k-1)k} a_{k-2} = \frac{-1}{\left(k + \left[-\frac{1}{2}\right] - \frac{1}{2}\right)k} a_{k-2} = \frac{-1}{(k + r_2 - r_1)k} a_{k-2} \quad .$$

Knowing that the denominator in your recursion formulas should factor into either

$$k(k + r_1 - r_2) \quad \text{or} \quad (k + r_2 - r_1)k$$

should certainly simplify your factoring of that denominator! And if your denominator does not so factor, then you know you made an error in your computations. So it provides a partial error check.

It also leads us to our next topic.

Problems Possibly Arising in Step 8

In discussing step 8 of the basic Frobenius method, we indicated that there may be "problems" in finding a modified series solution corresponding to r_2. Well, take a careful look at the recursion formula obtained using r_2:

$$a_k = \frac{F(k, r_2, a_0, a_1, \ldots, a_{k-1})}{(k - [r_1 - r_2])k} \qquad \text{for} \quad k \geq \kappa$$

(remember, κ is some positive integer). See the problem? That's correct, the denominator may be zero for one of the k's. And it occurs if (and only if) r_1 and r_2 differ by some integer K greater than or equal to κ,

$$r_1 - r_2 = K \quad .$$

Then, when we attempt to compute a_K, we get

$$\begin{aligned} a_K &= \frac{F(K, r_2, a_0, a_1, \ldots, a_{k-1})}{(K - [r_1 - r_2])K} \\ &= \frac{F(K, r_2, a_0, a_1, \ldots, a_{k-1})}{(K - K)K} = \frac{F(K, r_2, a_0, a_1, \ldots, a_{k-1})}{0} \quad ! \end{aligned}$$

Precisely what we can say about a_K when this happens depends on whether the numerator in this expression is zero or not. To clarify matters slightly, let's rewrite the above using recursion relation (33.24b) which, in this case, reduces to

$$0 \cdot a_K = F(K, r_2, a_0, a_1, \ldots, a_{k-1}) \tag{33.25}$$

Now:

1. If $F(K, r_2, a_0, a_1, \ldots, a_{k-1}) \neq 0$, then the recursion formula "blows up" at $k = K$,

$$a_K = \frac{F(K, r_2, a_0, a_1, \ldots, a_{k-1})}{0} \quad ,$$

   More precisely, a_K must satisfy

$$0 \cdot a_K = F(K, r_2, a_0, a_1, \ldots, a_{k-1}) \neq 0 \quad ,$$

   which is impossible. Hence, it is impossible for our differential equation to have a solution of the form $(x - x_0)^{r_2} \sum_{k=0}^{\infty} a_k (x - x_0)^k$ with $a_0 \neq 0$.

2. If $F(K, r_2, a_0, a_1, \ldots, a_{k-1}) = 0$, then the recursion equation for a_K reduces to

$$a_K = \frac{0}{0} \quad ,$$

   an indeterminant expression telling us nothing about a_K. More precisely, (33.25) reduces to

$$0 \cdot a_K = 0 \quad ,$$

   which is valid for any value of a_K; that is, a_K is another arbitrary constant (in addition to a_0). Moreover, a careful analysis will confirm that, in this case, we will re-derive the modified power series solution corresponding to r_1 with a_K being the arbitrary constant in that series solution.

But why don't we worry about the denominator in the recursion formula obtained using r_1,

$$a_k = \frac{F(k, r_1, a_0, a_1, \ldots, a_{k-1})}{k(k + r_1 - r_2)} \qquad \text{for} \quad k \geq \kappa \quad ?$$

Because this denominator is zero only if $r_1 - r_2$ is some negative integer, contrary to our labeling convention of $r_1 \geq r_2$.

!►Example 33.9: *Consider*

$$2xy'' - 4y' - y = 0 \quad .$$

Multiplying this out by x we get it into the form

$$2x^2 y'' - 4xy' - xy = 0 \quad .$$

As noted in example 33.5 on page 33–10, this equation has one regular singular point, $x_0 = 0$. Setting

$$y = y(x) = x^r \sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} a_k x^{k+r}$$

and differentiating, we get (as before)

$$y' = \frac{d}{dx} \sum_{k=0}^{\infty} a_k x^{k+r} = \sum_{k=0}^{\infty} a_k (k+r) x^{k+r-1}$$

and

$$y'' = \frac{d^2}{dx^2} \sum_{k=0}^{\infty} a_k x^{k+r} = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1) x^{k+r-2} \quad .$$

Plugging these into our differential equation:

$$\begin{aligned}
0 &= 2x^2 y'' - 4xy' - xy \\
&= 2x^2 \sum_{k=0}^{\infty} a_k (k+r)(k+r-1) x^{k+r-2} - 4 \sum_{k=0}^{\infty} a_k (k+r) x^{k+r-1} - x \sum_{k=0}^{\infty} a_k x^{k+r} \\
&= \sum_{k=0}^{\infty} 2a_k (k+r)(k+r-1) x^{k+r} - \sum_{k=0}^{\infty} 4a_k (k+r) x^{k+r} - \sum_{k=0}^{\infty} a_k x^{k+r+1} \quad .
\end{aligned}$$

Dividing out the x^r, reindexing, and grouping like terms:

$$\begin{aligned}
0 &= \underbrace{\sum_{k=0}^{\infty} 2a_k (k+r)(k+r-1) x^k}_{n = k} - \underbrace{\sum_{k=0}^{\infty} 4a_k (k+r) x^k}_{n = k} - \underbrace{\sum_{k=0}^{\infty} a_k x^{k+1}}_{n = k+1} \\
&= \sum_{n=0}^{\infty} 2a_n (n+r)(n+r-1) x^n - \sum_{n=0}^{\infty} 4a_n (n+r) x^n - \sum_{n=1}^{\infty} a_n x^n
\end{aligned}$$

$$= 2a_0(0+r)(0+r-1)x^0 + \sum_{n=1}^{\infty} 2a_n(n+r)(n+r-1)x^n$$

$$- 4a_0(0+r)x^0 - \sum_{n=1}^{\infty} 4a_{n-1}(n+r)x^n - \sum_{n=1}^{\infty} a_{n-1}x^n$$

$$= a_0[\underbrace{2r(r-1)-4r}_{2r^2-2r-4r}]x^0 + \sum_{n=1}^{\infty}\{a_n[\underbrace{2(n+r)(n+r-1)-4(n+r)}_{2(n+r)(n+r-1-2)}] - a_{n-1}\}x^n \quad .$$

Thus, our differential equation reduces to

$$a_0\left[2\left(r^2-3r\right)\right]x^0 + \sum_{n=1}^{\infty}\{a_n\left[2(n+r)(n+r-3)\right] - a_{n-1}\}x^n = 0 \quad . \tag{33.26}$$

From the first term, we get the indicial equation:

$$\underbrace{2\left(r^2-3r\right)}_{2r(r-3)} = 0$$

So r *can equal* 0 *or* 3. *Since* $3 > 0$, *our convention has us identifying these solutions to the indicial equation as*

$$r_1 = 3 \qquad \text{and} \qquad r_2 = 0 \quad .$$

Letting $r = r_1 = 3$ *in equation (33.26), we see that*

$$0 = a_0\left[2\left(3^2-3\cdot 3\right)\right]x^0 + \sum_{n=1}^{\infty}\{a_n\left[2(n+3)(n+3-3)\right] - a_{n-1}\}x^n$$

$$= a_0\,[0]\,x^0 + \sum_{n=1}^{\infty}\{a_n\left[2(n+3)n\right] - a_{n-1}\}x^n$$

So, for $n \geq 1$,

$$a_n\left[2(n+3)n\right] - a_{n-1} = 0 \quad .$$

Solving for a_n, *we obtain the recursion formula*

$$a_n = \frac{1}{2n(n+3)}a_{n-1} \qquad \text{for} \quad n = 1, 2, 3, \ldots \quad .$$

Equivalently[12],

$$a_k = \frac{1}{2k(k+3)}a_{k-1} \qquad \text{for} \quad k = 1, 2, 3, \ldots \quad .$$

Applying this recursion formula:

$$a_1 = \frac{1}{2\cdot 1(1+3)}a_0 = \frac{1}{2\cdot 1(4)}a_0 \quad ,$$

$$a_2 = \frac{1}{2\cdot 2(2+3)}a_1 = \frac{1}{2\cdot 2(5)}\cdot\frac{1}{2\cdot 1(4)}a_0 = \frac{1}{2^2(2\cdot 1)(5\cdot 4)}a_0$$

$$= \frac{3\cdot 2\cdot 1}{2^2(2\cdot 1)(5\cdot 4\cdot 3\cdot 2\cdot 1)}a_0 = \frac{6}{2^2(2\cdot 1)(5\cdot 4\cdot 3\cdot 2\cdot 1)}a_0 \quad ,$$

[12] Observe that this is formula (33.24a) with $\kappa = 1$ and $F(k, r_1, a_0, a_1, \ldots, a_{k-1}) = \frac{1}{2}a_{k-1}$.

$$a_3 = \frac{1}{2 \cdot 3(3+3)} a_2 = \frac{1}{2 \cdot 3(6)} \cdot \frac{6}{2^2(2 \cdot 1)(5 \cdot 4 \cdot 3 \cdot 2 \cdot 1)} a_0$$
$$= \frac{6}{2^3(3 \cdot 2 \cdot 1)(6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1)} a_0 \quad ,$$
$$\vdots$$
$$a_k = \frac{6}{2^k\, k!(k+3)!} a_0 \quad .$$

Notice that this last formula even holds for $k = 0$ *and* $k = 1$ *. So*

$$a_k = \frac{6}{2^k\, k!(k+3)!} a_0 \qquad \text{for} \quad k = 0, 1, 2, 3, \ldots \quad ,$$

and the corresponding modified power series solution is

$$y(x) = x^{r_1} \sum_{k=0}^{\infty} a_k x^k = a_0 x^3 \sum_{k=0}^{\infty} \frac{6}{2^k\, k!(k+3)!} x^k \quad .$$

Now let's try to find a similar solution corresponding to $r_2 = 0$ *. Letting* $r = r_2 = 0$ *in equation (33.26) yields*

$$0 = a_0 \left[2\left(0^2 - 3 \cdot 0\right)\right] x^0 + \sum_{n=1}^{\infty} \{a_n [2(n+0)(n+0-3)] - a_{n-1}\} x^n$$
$$= a_0 [0] x^0 + \sum_{n=1}^{\infty} \{a_n [2n(n-3)] - a_{n-1}\} x^n \quad .$$

So, for $n \geq 1$ *,*

$$a_n [2n(n-3)] - a_{n-1} = 0 \quad .$$

Solving for a_n *, we obtain the recursion formula*

$$a_n = \frac{1}{2n(n-3)} a_{n-1} \qquad \text{for} \quad n = 1, 2, 3, \ldots \quad .$$

Applying this, we get

$$a_1 = \frac{1}{2 \cdot 1(1-3)} a_0 = \frac{1}{2 \cdot 1(-2)} a_0 \quad ,$$
$$a_2 = \frac{1}{2 \cdot 2(2-3)} a_1 = \frac{1}{2 \cdot 2(-1)} \cdot \frac{1}{2 \cdot 1(-2)} a_0 = \frac{1}{2^2 2 \cdot 1(-1)(-2)} a_0 \quad ,$$
$$a_3 = \frac{1}{2 \cdot 3(3-3)} a_1 = \frac{1}{2 \cdot 2(0)} \cdot \frac{1}{2^2 2 \cdot 1(-1)(-2)} a_0 = \frac{1}{0} a_0 \quad !$$

In this case, the a_3 *"blows up" if* $a_0 \neq 0$ *. In other words, for this last equation to be true, we must have*

$$0 \cdot a_3 = a_0 \quad ,$$

contrary to our requirement that $a_0 \neq 0$ *. So, for this differential equation, our search for a solution of the form*

$$x^r \sum_{k=0}^{\infty} a_k x^k \qquad \text{with} \quad a_0 \neq 0$$

is doomed to failure when $r = r_2$. There is no such solution.

This leaves us with just the solutions found corresponding to $r = r_1$,

$$y(x) = a_0 x^3 \sum_{k=0}^{\infty} \frac{6}{2^k \, k!(k+3)!} x^k \quad .$$

In the next chapter, we will discuss how to find the complete general solution when, as in the last example, the basic method of Frobenius does not yield a second solution. We'll also finish solving the differential equation in the last example.

33.6 Dealing with Complex Exponents

In practice, the exponents r_1 and r_2 are usually real numbers. Still, complex exponents are possible. Fortunately, the differential equations arising from "real-world" problems usually have real-valued coefficients, which means that, if r_1 and r_2 are not real valued, they will probably be a conjugate pair

$$r_1 = r_+ = \lambda + i\omega \qquad \text{and} \qquad r_2 = r_- = \lambda - i\omega$$

with $\omega \neq 0$. In this case, step 7 of the Frobenius method (with $r_1 = r_+$, and renaming a_0 as c_+) leads to a solution of the form

$$y(x) = c_+ y_+(x)$$

where

$$y_+(x) = (x - x_0)^{\lambda + i\omega} \sum_{k=0}^{\infty} \alpha_k (x - x_0)^k$$

with $\alpha_0 = 1$. The other α_k's will be determined from the recursion formula and will simply be the a_k's described in the method with a_0 factored out. And they will probably be complex-valued since the recursion formula probably involves $r_+ = \lambda + i\omega$.

We could then continue with step 8 and compute the general series solution corresponding to $r_2 = r_-$. Fortunately, this lengthy set of computations is completely unnecessary because you can easily confirm that the complex conjugate y^* of any solution y to our differential equation is another solution to our differential equation (see exercise 33.7 on page 33–38). Thus, as a "second solution", we can just use

$$y_-(x) = \left[y_+(x)\right]^* = \left[(x - x_0)^{\lambda + i\omega} \sum_{k=0}^{\infty} \alpha_k (x - x_0)^k\right]^*$$

$$= (x - x_0)^{\lambda - i\omega} \sum_{k=0}^{\infty} {\alpha_k}^* (x - x_0)^k \quad .$$

It is also easy to verify that $\{y_+(x), y_-(x)\}$ is linearly independent. Hence,

$$y(x) = c_+ y_+(x) + c_- y_-(x)$$

is a general solution to our differential equation.

However, the two solutions, y_+ and y_- will be complex valued. To avoid complex-valued solutions, let's employ the trick used before to derive a corresponding linearly independent pair $\{y_1, y_2\}$ of real-valued solutions to our differential equation by setting

$$y_1(x) = \frac{1}{2}\left[y_+(x) + y_-(x)\right] \quad \text{and} \quad y_2(x) = \frac{1}{2i}\left[y_+(x) - y_-(x)\right] \quad .$$

Using the fact that

$$X^{\lambda \pm i\omega} = X^\lambda X^{i\omega} = X^\lambda\left[\cos(\omega \ln|X|) \pm i \sin(\omega \ln|X|)\right] \quad ,$$

it is a simple exercise to verify that

$$y_1(x) = (x - x_0)^\lambda \cos(\omega \ln|x - x_0|) \sum_{k=0}^{\infty} b_k (x - x_0)^k$$

and

$$y_2(x) = (x - x_0)^\lambda \sin(\omega \ln|x - x_0|) \sum_{k=0}^{\infty} c_k (x - x_0)^k$$

where the b_k's and c_k's are real-valued constants related to the above α_k's by

$$b_k = \frac{\alpha_k + \alpha_k{}^*}{2} \quad \text{and} \quad c_k = \frac{\alpha_k - \alpha_k{}^*}{2i} \quad .$$

33.7 Appendix: On Tests for Regular Singular Points

Proof of Theorem 33.2

The validity of theorem 33.2 follows immediately from lemma 33.1 (which we've already proven) and the next lemma.

Lemma 33.4

Let a, b and c be rational functions, and assume x_0 is a singular point on the real line for

$$a(x)y'' + b(x)y' + c(x)y = 0 \quad . \tag{33.27}$$

If the two limits

$$\lim_{x \to x_0} (x - x_0)\frac{b(x)}{a(x)} \quad \text{and} \quad \lim_{x \to x_0} (x - x_0)^2 \frac{c(x)}{a(x)}$$

are both finite values, then the differential equation can be written in quasi-Euler form

$$(x - x_0)^2 \alpha(x) y'' + (x - x_0)\beta(x) y' + \gamma(x) y = 0$$

where α, β and γ are polynomials with $\alpha(x_0) \neq 0$.

PROOF: Because a, b and c are rational functions (i.e., quotients of polynomials), we can multiply equation (33.27) through by the common denominators, obtaining a differential equation in which all the coefficients are polynomials. Then factoring out the $(x - x_0)$-factors, and dividing out the factors common to all terms, we obtain

$$(x - x_0)^k A(x)y'' + (x - x_0)^m B(x)y' + (x - x_0)^n C(x)y = 0 \tag{33.28}$$

where A, B an C are polynomials with

$$A(x_0) \neq 0 \quad , \quad B(x_0) \neq 0 \quad \text{and} \quad C(x_0) \neq 0 \quad ,$$

and k, m and n are nonnegative integers, one of which must be zero. Moreover, since this is just a rewrite of equation (33.27), and x_0 is, by assumption, a singular point for this differential equation, we must have that $k \geq 1$. Hence $m = 0$ or $n = 0$.

Dividing equations (33.27) and (33.28) by their leading terms then yields, respectively,

$$y'' + \frac{b(x)}{a(x)}y' + \frac{c(x)}{a(x)}y = 0$$

and

$$y'' + (x - x_0)^{m-k}\frac{B(x)}{A(x)}y' + (x - x_0)^{n-k}\frac{C(x)}{A(x)}y = 0 \quad .$$

But these last two equations describe the same differential equation, and have the same first coefficients. Consequently, the other coefficients must be the same, giving us

$$\frac{b(x)}{a(x)} = (x - x_0)^{m-k}\frac{B(x)}{A(x)} \quad \text{and} \quad \frac{c(x)}{a(x)} = (x - x_0)^{n-k}\frac{C(x)}{A(x)} \quad .$$

Thus,

$$\lim_{x \to x_0}(x - x_0)\frac{b(x)}{a(x)} = \lim_{x \to x_0}(x - x_0)^{m+1-k}\frac{B(x)}{A(x)} = \lim_{x \to x_0}(x - x_0)^{m+1-k}\frac{B(x_0)}{A(x_0)}$$

and

$$\lim_{x \to x_0}(x - x_0)^2\frac{c(x)}{a(x)} = \lim_{x \to x_0}(x - x_0)^{n+2-k}\frac{C(x)}{A(x)} = \lim_{x \to x_0}(x - x_0)^{n+2-k}\frac{C(x_0)}{A(x_0)} \quad .$$

By assumption, though, these limits are finite, and for the limits on the right to be finite, the exponents must be zero or larger. This, combined with the fact that $k \geq 1$ means that k, m and l must satisfy

$$m + 1 \geq k \geq 1 \quad \text{and} \quad n + 2 \geq k \geq 1 \quad . \tag{33.29}$$

But remember, m, n and k are nonnegative integers with $m = 0$ or $n = 0$. That leaves us with only three possibilities:

1. If $m = 0$, then the first inequality in set (33.29) reduces to

$$1 \geq k \geq 1$$

telling us that $k = 1$. Equation (33.28) then becomes

$$(x - x_0)^1 A(x)y'' + B(x)y' + (x - x_0)^n C(x)y = 0 \quad .$$

Muliplying through by $x - x_0$ this becomes

$$(x - x_0)^2\alpha(x)y'' + (x - x_0)\beta(x)y' + \gamma(x)y = 0$$

where α, β and γ are the polynomials

$$\alpha(x) = A(x) \quad , \quad \beta(x) = B(x) \quad \text{and} \quad \gamma(x) = (x - x_0)^{n+1}C(x) \quad .$$

2. If $n = 0$, then the second inequality in set (33.29) reduces to

$$2 \geq k \geq 1 \quad .$$

So $k = 1$ or $k = 2$, giving us two options:

(a) If $k = 1$, then equation (33.28) becomes

$$(x - x_0)^1 A(x)y'' + (x - x_0)^m B(x)y' + C(x)y = 0 \quad ,$$

which can be rewritten as

$$(x - x_0)^2 \alpha(x)y'' + (x - x_0)\beta(x)y' + \gamma(x)y = 0$$

where α, β and γ are the polynomials

$$\alpha(x) = A(x) \quad , \quad \beta(x) = (x-x_0)^m B(x) \quad \text{and} \quad \gamma(x) = (x-x_0)C(x) \quad .$$

(b) If $k = 2$, then the other inequality inequality in set (33.29) becomes

$$m + 1 \geq 2$$

Hence $m \geq 1$. In this case, equation (33.28) becomes

$$(x - x_0)^2 A(x)y'' + (x - x_0)^m B(x)y' + C(x)y = 0$$

which is the same as

$$(x - x_0)^2 \alpha(x)y'' + (x - x_0)\beta(x)y' + \gamma(x)y = 0$$

where α, β and γ are the polynomials

$$\alpha(x) = A(x) \quad , \quad \beta(x) = (x-x_0)^{m-1} B(x) \quad \text{and} \quad \gamma(x) = C(x) \quad .$$

Note that, in each case, we can rewrite our differential equation as

$$(x - x_0)^2 \alpha(x)y'' + (x - x_0)\beta(x)y' + \gamma(x)y = 0$$

where α, β and γ are polynomials with

$$\alpha(x_0) = \alpha_0(x_0) \neq 0 \quad .$$

So, in each case, we have rewritten our differential equation in quasi-Euler form, as claimed in the lemma. ▌

Testing for Regularity When the Coefficients Are Not Rational

By using lemma 30.13 on page 30–21 on factoring analytic functions, or lemma 30.14 on page 30–22 on factoring quotients of analytic functions, you can easily modify the arguments in the last proof to obtain an analog of the above lemma appropriate to differential equations whose coefficients are not rational. Basically, just replace

"rational functions" with "quotients of functions analytic at x_0" ,

and replace

"polynomials" with "functions analytic at x_0" .

The resulting lemma, along with lemma 33.1, then yields the following analog to theorem 33.2.

Theorem 33.5 (testing for regular singular points (ver.2))
Assume x_0 is a singular point on the real line for the differential equation

$$a(x)y'' + b(x)y' + c(x)y = 0$$

where a, b and c are quotients of functions analytic at x_0. Then x_0 is a regular singular point for this differential equation if and only if the two limits

$$\lim_{x\to x_0}(x-x_0)\frac{b(x)}{a(x)} \qquad \text{and} \qquad \lim_{x\to x_0}(x-x_0)^2\frac{c(x)}{a(x)}$$

are both finite values.

Additional Exercises

33.2. *Find a fundamental set of solutions $\{y_1, y_2\}$ for each of the following shifted Euler equations, and sketch the graphs of y_1 and y_2 near the singular point.*

a. $(x-3)^2y'' - 2(x-3)y' + 2y = 0$

b. $(x-1)^2y'' - 5(x-1)y' + 9y = 0$

c. $2x^2y'' + 5xy' + 1y = 0$

d. $(x+2)^2y'' + (x+2)y' = 0$

e. $3(x-5)^2y'' - 4(x-5)y' + 2y = 0$

f. $(x-5)^2y'' + (x-5)y' + 4y = 0$

33.3. *Identify all of the singular points for each of the following differential equations, and determine which of those on the real line are regular singular points, and which are irregular singular points. Also, find the Frobenius radius of convergence R for the given differential equation about the given x_0.*

a. $x^2y'' + \dfrac{x}{x-2}y' + \dfrac{2}{x+2}y = 0 \quad , \quad x_0 = 0$

b. $x^3y'' + x^2y' + y = 0 \quad , \quad x_0 = 2$

c. $\left(x^3 - x^4\right)y'' + (3x-1)y' + 827y = 0 \quad , \quad x_0 = 1$

d. $y'' + \frac{1}{x-3}y' + \frac{1}{x-4}y = 0 \quad , \quad x_0 = 3$

e. $y'' + \frac{1}{(x-3)^2}y' + \frac{1}{(x-4)^2}y = 0 \quad , \quad x_0 = 4$

f. $y'' + \left(\frac{1}{x} - \frac{1}{3}\right)y' + \left(\frac{1}{x} - \frac{1}{4}\right)y = 0 \quad , \quad x_0 = 0$

g. $\left(4x^2 - 1\right)y'' + \left(4 - \frac{2}{x}\right)y' + \frac{1-x^2}{1+x^2}y = 0 \quad , \quad x_0 = 0$

h. $\left(4 + x^2\right)^2 y'' + y = 0 \quad , \quad x_0 = 0$

33.4. *For each of the following differential equations, verify that* $x_0 = 0$ *is a regular singular point, and then use the basic method of Frobenius to find modified power series solutions. In particular:*

- i *Find and solve the corresponding indicial equation for the equation's exponents* r_1 *and* r_2 *.*
- ii *Find the recursion formula corresponding to each exponent.*
- iii *Find and explicitly write out at least the first four nonzero terms of all series solutions about* $x_0 = 0$ *that can be found by the basic Frobenius method (if a series terminates, find all the nonzero terms).*
- iv *Try to find a general formula for all the coefficients in each series.*
- v *When a second particular solution cannot be found by the basic method, give a reason that second solution cannot be found.*

a. $x^2y'' - 2xy' + \left(x^2 + 2\right)y = 0$

b. $4x^2y'' + (1 - 4x)y = 0$

c. $x^2y'' + xy' + (4x - 4)y = 0$

d. $\left(x^2 - 9x^4\right)y'' - 6xy' + 10y = 0$

e. $x^2y'' - xy' + \frac{1}{1-x}y = 0$

f. $y'' + \frac{1}{x}y' + y = 0$ *(Bessel's equation of order* 0 *)*

g. $y'' + \frac{1}{x}y' + \left[1 - \frac{1}{x^2}\right]y = 0$ *(Bessel's equation of order* 1 *)*

h. $2x^2y'' + \left(5x - 2x^3\right)y' + (1 - x^2)y = 0$

i. $x^2y'' - \left(5x + 2x^2\right)y' + (9 + 4x)y = 0$

j. $\left(3x^2 - 3x^3\right)y'' - \left(4x + 5x^2\right)y' + 2y = 0$

k. $x^2y'' - \left(x + x^2\right)y' + 4xy = 0$

l. $4x^2y'' + 8x^2y' + y = 0$

m. $x^2y'' + \left(x - x^4\right)y' + 3x^3y = 0$

n. $\left(9x^2 + 9x^3\right)y'' + \left(9x + 27x^2\right)y' + (8x - 1)y = 0$

33.5. *For each of the following, verify that the given* x_0 *is a regular singular point for the given differential equation, and then use the basic method of Frobenius to find modified power series solutions to that differential equation. In particular:*

- *i* *Find and solve the corresponding indicial equation for the equation's exponents* r_1 *and* r_2 *.*
- *ii* *Find the recursion formula corresponding to each exponent.*
- *iii* *Find and explicitly write out at least the first four nonzero terms of all series solutions about* $x_0 = 0$ *that can be found by the basic Frobenius method (if a series terminates, find all the nonzero terms).*
- *iv* *Try to find a general formula for all the coefficients in each series.*
- *v* *When a second particular solution cannot be found by the basic method, give a reason that second solution cannot be found.*

a. $(x - 3)y'' + (x - 3)y' + y = 0 \quad , \quad x_0 = 3$

b. $y'' + \dfrac{2}{x + 2}y' + y = 0 \quad , \quad x_0 = -2$

c. $4y'' + \dfrac{4x - 3}{(x - 1)^2}y = 0 \quad , \quad x_0 = 1$

d. $(x - 3)^2y'' + \left(x^2 - 3x\right)y' - 3y = 0 \quad , \quad x_0 = 3$

33.6. *Let* $\sum_{k=0}^{\infty} a_k x^k$ *be a power series convergent on* $(-R, R)$ *for some* $R > 0$ *. We already know that, on* $(-R, R)$ *,*

$$\frac{d}{dx}\sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} \frac{d}{dx}\left[a_k x^k\right] \quad .$$

a. *Using the above and the product rule confirm that, on* $(-R, R)$ *,*

$$\frac{d}{dx}\left[x^r \sum_{k=0}^{\infty} a_k x^k\right] = \sum_{k=0}^{\infty} a_k (k + r) x^{k+r-1} \quad .$$

for any real value r *.*

b. *How does this justify the term-by-term differentiation used in step 2 of the basic Frobenius method?*

33.7. *For the following, assume* u *and* v *are real-valued functions on some interval* $\mathcal{I}$ *, and let* $y = u + iv$ *.*

a. *Verify that*

$$\frac{dy^*}{dx} = \left(\frac{dy}{dx}\right)^* \qquad \text{and} \qquad \frac{d^2y^*}{dx^2} = \left(\frac{d^2y}{dx^2}\right)^*$$

where y^* *is the complex conjugate of* y *.*

b. *Further assume that, on some interval* $\mathcal{I}$ *,* y *satisfies*

$$a(x)\frac{d^2y}{dx^2} + b(x)\frac{dy}{dx} + c(x)y = 0$$

where a *,* b *and* c *are all real-valued functions. Using the results from the previous part of this exercise, show that*

$$a(x)\frac{d^2y^*}{dx^2} + b(x)\frac{dy^*}{dx} + c(x)y^* = 0 \quad .$$

Some Answers to Some of the Exercises

WARNING! Most of the following answers were prepared hastily and late at night. They have not been properly proofread! Errors are likely!

2a. $y_1(x) = (x-3)^2$, $y_2(x) = x - 3$
2b. $y_1(x) = (x-1)^3$, $y_2(x) = (x-1)^3 \ln|x-1|$
2c. $y_1(x) = x^{-1/2}$, $y_2(x) = x^{-1}$
2d. $y_1(x) = 1$, $y_2(x) = \ln|x+2|$
2e. $y_1(x) = (x-5)^2$, $y_1(x) = \sqrt[3]{x-5}$
2f. $y_1(x) = \cos(2\ln|x-5|)$, $y_1(x) = \sin(2\ln|x-5|)$
3a. Reg. sing. pts.: $0,\ 2,\ -2$; No irreg. sing. pt. ; $R = 2$
3b. No Reg. sing. pt. ; Irreg. sing. pt.: 0 ; $R = 2$
3c. Reg. sing. pt.: 1 ; Irreg. sing. pt.: 0 ; $R = 1$
3d. Reg. sing. pt.: $3,\ 4$; No irreg. sing. pt. ; $R = 1$
3e. Reg. sing. pt.: 4 ; Irreg. sing. pt.: 3 ; $R = 1$
3f. Reg. sing. pt.: 0 ; No irreg. sing. pt. ; $R = \infty$
3g. Reg. sing. pts.: $0,\ \frac{1}{2},\ -\frac{1}{2},\ i,\ -i$; No irreg. sing. pts. ; $R = \frac{1}{2}$
3h. Reg. sing. pts.: $2i,\ -2i$; No irreg. sing. pt. ; $R = 2$
4a. $r^2 - 3r + 2 = 0$; $r_1 = 2$, $r_2 = 1$;

For $r = r_1$, $a_1 = 0$, $a_k = \frac{-1}{(k+1)k}a_{k-2}$; $y(x) = ay_1(x)$ with

$$y_1(x) = x^2\left[1 - \frac{1}{3!}x^2 + \frac{1}{5!}x^4 - \frac{1}{7!}x^6 - \cdots\right] = x^2\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)!}x^{2m} = x\sin(x)$$

For $r = r_2$, $a_1 = 0$; $a_k = \frac{-1}{k(k-1)}a_{k-2}$; $y(x) = ay_2(x)$ with

$$y_2(x) = x\left[1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots\right] = x\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m)!}x^{2m} = x\cos(x)$$

4b. $4r^2 - 4r + 1 = 0$; $r_1 = r_2 = \frac{1}{2}$;

For $r = r_1$: $a_k = \frac{1}{k^2}a_{k-1}$; $y(x) = ay_1(x)$ with

$$y_1(x) = \sqrt{x}\sum_{k=0}^{\infty}\frac{1}{(k!)^2}x^k$$

No “second” value of r

4c. $r^2 - 4 = 0$; $r_1 = 2$, $r_2 = -2$;

For $r = r_1$: $a_k = \frac{-4}{k(k+4)}a_{k-1}$; $y(x) = ay_1(x)$ with

$$y_1(x) = x^2\left[1 - \frac{4}{5}x + \frac{4^2}{(2)(6\cdot 5)}x^2 - \frac{4^3}{(3\cdot 2)(7\cdot 6\cdot 5)}x^3 + \cdots\right] = x^2\sum_{k=0}^{\infty}\frac{(-4)^k 4!}{k!(k+4)!}x^k$$

For $r = r_2$, the recursion formula blows up

4d. $r^2 - 7r + 10 = 0$; $r_1 = 5$, $r_2 = 2$;

For $r = r_1$, $a_1 = 0$, $a_k = \frac{9(k+2)}{k}a_{k-2}$; $y(x) = ay_1(x)$ with

$$y_1(x) = x^2\left[1 + \frac{9\cdot 4}{2}x^2 + \frac{9^2\cdot 6}{2}x^4 + \frac{9^3\cdot 8}{2}x^6 - \cdots\right] = x^5\sum_{m=0}^{\infty}9^m(m+1)x^{2m}$$

For $r = r_2$, $a_1 = 0$; $a_k = \frac{9(k-1)}{k-3}a_{k-2}$; $y(x) = ay_2(x)$ with

$$y_2(x) = x^2\left[1 - 9x^2 - 3\cdot 9^2x^4 - 5\cdot 9^3x^6 + \cdots\right] = x^2\sum_{m=0}^{\infty} 9^m(1-2m)x^{2m}$$

4e. $r^2 - 2r + 1 = 0$; $r_1 = r_2 = 1$;
For $r = r_1$: $a_k = \frac{k-2}{k}a_{k-1}$; $y(x) = ay_1(x)$ with
$y_1(x) = x - x^2$
No "second" value of r

4f. $r^2 = 0$; $r_1 = r_2 = 0$;
For $r = r_1$: $a_1 = 0$, $a_k = \frac{-1}{k^2}a_{k-2}$; $y(x) = ay_1(x)$ with

$$y_1(x) = 1 - \frac{1}{2^2}x^2 + \frac{1}{(2\cdot 4)^2}x^4 - \frac{1}{(6\cdot 4\cdot 2)^2}x^6 + \cdots = \sum_{m=0}^{\infty}\frac{(-1)^m}{(2^m m!)^2}x^{2m}$$

No "second value" for r

4g. $r^2 - 1 = 0$; $r_1 = 1$, $r_2 = -1$;
For $r = r_1$: $a_1 = 0$, $a_k = \frac{-1}{k(k+2)}a_{k-2}$; $y(x) = ay_1(x)$ with

$$y_1(x) = x\left[1 - \frac{1}{2\cdot 4}x^2 + \frac{1}{(4\cdot 2)(6\cdot 4)}x^4 - \frac{1}{(6\cdot 4\cdot 2)(8\cdot 6\cdot 4)}x^6 + \cdots\right] = x\sum_{m=0}^{\infty}\frac{(-1)^m}{2^{2m}m!(m+1)!}x^{2m}$$

For $r = r_2$, the recursion formula blows up

4h. $2r^2 + 3r + 1 = 0$; $r_1 = -1/2$, $r_2 = -1$;
For $r = r_1$: $a_1 = 0$, $a_k = \frac{2(k-2)}{k(2k+1)}a_{k-2}$; $y(x) = ay_1(x)$ with
$y_1(x) = x^{-1/2}$
For $r = r_2$: $a_1 = 0$, $a_k = \frac{2k-5}{k(2k-1)}a_{k-2}$; $y(x) = ay_2(x)$ with

$$y_2(x) = x^{-1}\left[1 - \frac{1}{2\cdot 3}x^2 - \frac{1}{4\cdot 2\cdot 7}x^4 - \frac{1}{6\cdot 4\cdot 2\cdot 11}x^6 + \cdots\right] = x^{-1}\sum_{m=0}^{\infty}\frac{-1}{2^m m!(4m-1)}x^{2m}$$

4i. $r^2 - 6r + 9 = 0$; $r_1 = r_2 = 3$;
For $r = r_1$: $a_k = \frac{2}{k}a_{k-1}$; $y(x) = ay_1(x)$ with

$$y_1(x) = x^3\left[1 + \frac{2}{1}x + \frac{2^2}{2}x^2 + \frac{2^3}{3\cdot 2}x^3 + \cdots\right] = x^3\sum_{k=0}^{\infty}\frac{2^k}{k!}x^k = x^3e^{2x}$$

No "second" value of r

4j. $3r^2 - 7r + 2 = 0$; $r_1 = 2$, $r_2 = 1/3$;
For $r = r_1$: $a_k = \frac{k+1}{k}a_{k-1}$; $y(x) = ay_1(x)$ with

$$y_1(x) = x^2\left[1 + 2x + 3x^2 + 4x^3 + \cdots\right] = x^2\sum_{k=0}^{\infty}(k+1)x^k$$

For $r = r_2$: $a_k = \frac{3k-2}{3k-5}a_{k-1}$; $y(x) = ay_1(x)$ with

$$y_1(x) = \sqrt[3]{x}\left[1 - \frac{1}{2}x - \frac{4}{2}x^2 - \frac{7}{2}x^3 + \cdots\right] = \sqrt[3]{x}\sum_{k=0}^{\infty}\frac{2-3k}{2}x^k$$

4k. $r^2 - 2r = 0$; $r_1 = 2$, $r_2 = 0$;
For $r = r_1$: $a_k = \frac{k-3}{k(k+2)}a_{k-1}$; $y(x) = ay_1(x)$ with
$y_1(x) = x^2 - \frac{2}{3}x^3 + \frac{1}{12}x^4$
For $r = r_2$, the recursion formula blows up

4l. $4r^2 - 4r + 1 = 0$; $r_1 = r_2 = 1/2$;

For $r = r_1$: $a_k = -\frac{2k-1}{k^2}a_{k-1}$; $y(x) = ay_1(x)$ with

$$y_1(x) = \sqrt{x}\left[1 - x + \frac{3}{4}x^2 - \frac{5\cdot 3}{9\cdot 4}x^3 + \cdots\right]$$

$$= \sqrt{x}\left[1 + \sum_{k=1}^{\infty}(-1)^k \frac{(2k-1)(2k-3)\cdots 5\cdot 3\cdot 1}{(k!)^2}x^k\right]$$

No "second" value of r

4m. $r^2 = 0$; $r_1 = r_2 = 0$;

For $r = r_1$: $a_1 = 0$, $a_2 = 0$, $a_k = \frac{k-6}{k^2}a_{k-3}$; $y(x) = ay_1(x)$ with

$$y_1(x) = 1 - \frac{1}{3}x^3$$

No "second" value of r

4n. $9r^2 - 1 = 0$; $r_1 = 1/3$, $r_2 = -1/3$;

For $r = r_1$: $a_k = -a_{k-1}$; $y(x) = ay_1(x)$ with

$$y_1(x) = x^{1/3}\left[1 - x + x^2 - x^3 + \cdots\right] = x^{1/3}\sum_{k=0}^{\infty}(-1)^k x^k = x^{1/3}(1+x)^{-1}$$

For $r = r_2$: $a_k = -a_{k-1}$; $y(x) = ay_2(x)$ with

$$y_2(x) = x^{-1/3}\left[1 - x + x^2 - x^3 + \cdots\right] = x^{-1/3}\sum_{k=0}^{\infty}(-1)^k x^k = x^{-1/3}(1+x)^{-1}$$

5a. $r^2 - r = 0$; $r_1 = 1$, $r_2 = 0$;

For $r = r_1$: $a_k = -\frac{1}{k}a_{k-1}$; $y(x) = ay_1(x)$ with

$$y_1(x) = (x-3)\left[1 - \frac{1}{2}(x-3) + \frac{1}{2\cdot 1}(x-3)^2 - \frac{1}{3\cdot 2\cdot 1}(x-3)^3 + \cdots\right]$$

$$= (x-3)\sum_{k=0}^{\infty}(-1)^k\frac{1}{k!}(x-3)^k = (x-3)e^{-(x-3)}$$

For $r = r_2$, the recursion formula blows up

5b. $r^2 + r = 0$; $r_1 = 0$, $r_2 = -1$;

For $r = r_1$: $a_k = -\frac{1}{(k+1)k}a_{k-2}$; $y(x) = ay_1(x)$ with

$$y_1(x) = 1 - \frac{1}{3!}(x+2)^2 + \frac{1}{5!}(x+2)^4 - \frac{1}{7!}(x+2)^6 + \cdots = \frac{\sin(x+2)}{x+2}$$

For $r = r_2$: $a_k = -\frac{1}{k(k-1)}a_{k-2}$; $y(x) = ay_2(x)$ with

$$y_2(x) = \frac{1}{x+2}\left[1 - \frac{1}{2!}(x+2)^2 + \frac{1}{4!}(x+2)^4 - \frac{1}{6!}(x+2)^6 + \cdots\right] = \frac{\cos(x+2)}{x+2}$$

5c. $4r^2 - 4r + 1 = 0$; $r_1 = r_2 = \frac{1}{2}$;

For $r = r_1$: $a_k = -\frac{1}{k^2}a_{k-1}$; $y(x) = ay_1(x)$ with

$$y_1(x) = \sqrt{x-1}\sum_{k=0}^{\infty}(-1)^k\frac{1}{(k!)^2}(x-1)^k$$

No "second" value of r

5d. $r^2 + 2r - 3 = 0$; $r_1 = 1$, $r_2 = -3$;

For $r = r_1$: $a_k = \frac{-1}{k+4} a_{k-1}$; $y(x) = ay_1(x)$ with

$$y_1(x) = (x-3)\left[1 - \frac{1}{5}(x-3) + \frac{1}{6\cdot 5}(x-3)^2 - \frac{1}{7\cdot 6\cdot 5}(x-3)^3 + \cdots\right]$$

$$= (x-3)\sum_{k=0}^{\infty} \frac{(-1)^k 4!}{(k+4)!}(x-3)^k$$

For $r = r_2$: $a_k = -\frac{1}{k} a_{k-1}$; $y(x) = ay_2(x)$ with

$$y_2(x) = (x-3)^{-3}\left[1 - (x-3) + \frac{1}{2}(x-3)^2 - \frac{1}{3\cdot 2}(x-3)^3 + \cdots\right]$$

$$= (x-3)^{-3}\sum_{k=0}^{\infty} \frac{(-1)^k}{k!}(x-3)^k = (x-3)^{-3} e^{-(x-3)}$$